Chapter 9: Integer Programming
Learning Objectives (see p. 382)

- **After completing this chapter, you should be able to:**
  - How Integer Programming (IP) differs from LP
  - Describe how general IP problems arise
  - Formulate a general IP model from a description of the problem
  - Describe how binary decision variables are used to represent yes-or-no decisions
  - Use binary decision variables to formulate constraints for mutually exclusive alternatives and contingent decisions
  - Formulate a binary integer programming model from a description of the problem
  - List some areas where important applications of binary integer programming are occurring
What Integer Programming means?

• Integer Programming (IP) is another important management science technique that is closely related to LP

• The only difference between LP and IP is that IP require INTEGER SOLUTIONS

• Thus, compared to LP, IP has the additional restriction that some or all of the decisions variables must have integer values (0, 1, 2, …)

• Integer programming problems falls into two categories:
  – General integer programming allows at least some of the integer variables to have any integer values satisfying the functional constraints and nonnegativity constraints
  – Binary integer programming restricts the integer variables to be Binary variables which are variables whose only possible values are 0 and 1 (yes-or-no decisions).
The TBA Airlines Problem

• TBA Airlines is a small regional company that specializes in short flights in small airplanes.

• The company has been doing well and has decided to expand its operations.

• The basic issue facing management is whether to purchase more small airplanes to add some new short flights, or start moving into the national market by purchasing some large airplanes, or both.

Question: How many airplanes of each type should be purchased to maximize their total net annual profit?
Data for the TBA Airlines Problem

<table>
<thead>
<tr>
<th></th>
<th>Small Airplane</th>
<th>Large Airplane</th>
<th>Capital Available</th>
</tr>
</thead>
<tbody>
<tr>
<td>Net annual profit per airplane</td>
<td>$1 million</td>
<td>$5 million</td>
<td></td>
</tr>
<tr>
<td>Purchase cost per airplane</td>
<td>5 million</td>
<td>50 million</td>
<td>$100 million</td>
</tr>
<tr>
<td>Maximum purchase quantity</td>
<td>2</td>
<td>No maximum</td>
<td></td>
</tr>
</tbody>
</table>

Before turning to a spreadsheet, we will solve this problem graphically in order to compare between LP and IP results
Linear Programming Formulation

Let \( S = \) Number of small airplanes to purchase

\( L = \) Number of large airplanes to purchase

Maximize Profit = \( S + 5L \) ($millions)

subject to

Capital Available: \( 5S + 50L \leq 100 \) ($millions)

Max Small Planes: \( S \leq 2 \)

and

\( S \geq 0, \ L \geq 0. \)
Graphical Method for Linear Programming

Number of large airplanes purchased

Number of small airplanes purchased

Feasible region

Profit = 11 = S + 5 L

(2, 1.8) = Optimal solution

(2, 1) = Rounded solution
(Profit = 7)

Rounding an OLPS may not give an OIPS
Violates Divisibility Assumption of LP

• **Divisibility Assumption of Linear Programming**: Decision variables in a linear programming model are allowed to have *any* values, including *fractional* values, that satisfy the functional and nonnegativity constraints. Thus, these variables are *not* restricted to just integer values.

• Since the number of airplanes purchased by TBA must have an integer value, the divisibility assumption is violated.
Integer Programming Formulation

Let $S =$ Number of small airplanes to purchase
$L =$ Number of large airplanes to purchase

\[
\text{Maximize Profit} = S + 5L \text{ ($millions$)}
\]
subject to
\[
\text{Capital Available: } 5S + 50L \leq 100 \text{ ($millions$)}
\]
Max Small Planes: $S \leq 2$

and

\[
S \geq 0, \quad L \geq 0
\]

\[
S, L \text{ are integers.}
\]

The LP relaxation of any IP problem is the original LP without the additional constraints that make some variables having integer values
Graphical Method for Integer Programming

• When an integer programming problem has just two decision variables, its optimal solution can be found by applying the graphical method for linear programming with just one change at the end.

• We begin as usual by graphing the feasible region for the LP relaxation, determining the slope of the objective function lines, and moving a straight edge with this slope through this feasible region in the direction of improving values of the objective function.

• However, rather than stopping at the last instant the straight edge passes through this feasible region, we now stop at the last instant the straight edge passes through an integer point that lies within this feasible region.

• This integer point is the optimal solution.
Graphical Method for Integer Programming

(0, 2) = Optimal solution for the integer programming problem (Profit = 10)

(2, 1.8) = Optimal solution for the LP relaxation (Profit = 11)

Profit = 10 = S + 5 L

(2, 1) = Rounded solution (Profit = 7)
### Spreadsheet Model

<table>
<thead>
<tr>
<th></th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>Small Airplane</td>
<td>Large Airplane</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Unit Profit ($millions)</td>
<td>1</td>
<td>5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td>Capital</td>
<td>Capital</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td>Capital Per Unit Produced</td>
<td>Spent</td>
<td>Available</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>Capital ($millions)</td>
<td>5</td>
<td>50</td>
<td>100</td>
<td></td>
<td>100</td>
</tr>
<tr>
<td>9</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Total Profit</td>
</tr>
<tr>
<td>11</td>
<td>Small Airplane</td>
<td>Large Airplane</td>
<td>($millions)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>Units Produced</td>
<td>0</td>
<td>2</td>
<td></td>
<td></td>
<td>10</td>
</tr>
<tr>
<td>13</td>
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<td>&lt;=</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>Maximum Small Airplanes</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Types of Integer Programming Problems

- Pure integer programming problems are those where all the decision variables must be integers.

- Mixed integer programming problems only require some of the variables (the “integer variables”) to have integer values so the divisibility assumption holds for the rest (the “continuous variables”).

- Binary integer programming (BIP) problems are those where all the decision variables restricted to integer values are further restricted to be binary variables.
  - Such problems can be further characterized as either pure BIP problems or mixed BIP problems, depending on whether all the decision variables or only some of them are binary variables.
Applications of Binary Variables

• Since binary variables only provide two choices, they are ideally suited to be the decision variables when dealing with **yes-or-no decisions**.

• Examples:
  – Should we undertake a particular fixed project?
  – Should we make a particular fixed investment?
  – Should we locate a facility in a particular site?
Some Other Applications

• Investment Analysis
  – Should we make a certain fixed investment?

• Site Selection
  – Should a certain site be selected for the location of a new facility?
  – Example: AT&T (1990)

• Designing a Production and Distribution Network
  – Should a certain plant remain open? Should a certain site be selected for a new plant? Should a distribution center remain open? Should a certain site be selected for a new distribution center? Should a certain distribution center be assigned to serve a certain market area?

All references available for download at www.mhhe.com/hillier2e/articles
Some Other Applications

• Dispatching Shipments
  – Should a certain route be selected for a truck? Should a certain size truck be used? Should a certain time period for departure be used?

• Scheduling Interrelated Activities
  – Should a certain activity begin in a certain time period?

• Scheduling Asset Divestitures
  – Should a certain asset be sold in a certain time period?

• Airline Applications:
  – Should a certain type of airplane be assigned to a certain flight leg? Should a certain sequence of flight legs be assigned to a crew?

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