Area Under the Normal Distribution

Summary about the Normal distribution:

- Is a continuous distribution.
- It is a bell shape distribution.
- It is symmetric distribution around the Mean ($\mu$).
- Runs from $-\infty$ to $+\infty$
- The shape of distribution depends on two parameters:
  \[ \mu = \text{Mean}, \quad \sigma = \text{Standard Deviation}. \]
- The total area under the distribution (PDF) equals 1.
- Any proportion of the (PDF) represents the probability of an event.
The Probability Density Function of the normal distribution (PDF) is:

\[ f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < +\infty \]
Standard Normal Distribution

- Is a special case of the Normal distribution Formed when the mean $\mu = 0$ and the standard deviation $\sigma = 1$.
- The probability density function of the Standard Normal distribution has a symmetric Bell shaped curve that is symmetric around the 0 (mean).
- The probability density function of the standard normal distribution is:

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \quad -\infty < z < +\infty$$

The Area under the Standard Normal Distribution

If $Z$ is a random variable that follows the standard normal distribution. That is, $Z \sim N(0,1)$. Then what is the probability that $Z$ will have a value between 0.5 and 2.2?

To find the $P(0.5 \leq Z \leq 2.2)$ which is the area under the standard normal curve from $Z$ equals 0.5 to $Z=2.2$. One can integrate the probability density function of the standard normal from $Z = 0.5$ to $Z = 2.2$

$$\int_{0.5}^{2.2} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$
The integration is not straight forward.

That is why a table was developed to find out any area under the standard normal distribution.

The table gives the area under the standard normal curve from \( z = 0 \) to any positive value of \( Z \). (practically from 0 to 4 standard deviations. As the standard deviation of the standard normal distribution equals 1).

We have to find out the integer value and the first decimal number of the positive value of \( Z \) from the first column of the table (The \( Z \) column).

Then specify the right second decimal column for the \( Z \) value.

The interaction of the row of \( Z \) value and the second decimal column for it is the cell that gives the required probability or the area under the standard normal curve.
### Area Under Standard Normal Distribution

The Area under the Standard Normal Distribution $P(0 < Z < z) = 1 - 0.5 = 0.5$.
The Area under the Standard Normal Distribution

(4) \[ P(0 < Z < 1.25) = \]

\[ (\text{Here } Z = 1.25 \text{ i.e. } 1.2 + 0.05) \]

\[ = 0.3944 \text{ OR } 39.44\% \]

\[ P(0 < Z < 0.96) \]

\[ (\text{Here } Z = 0.96 \text{ i.e. } 0.9 + 0.06) \]

\[ = 0.3315 \text{ OR } 33.15\% \]

(5) \[ P(Z > 1.37) = \]

\[ = 0.5 - P(0 < Z < 1.37) \]

\[ = 0.5 - 0.4147 = 0.0853 \text{ OR } 8.53\% \]

The Area under the Standard Normal Distribution

(6) \[ P(Z \geq 2.13) = 0.5 - P(0 \leq Z \leq 2.13) \]

\[ = 0.5 - 0.4834 - 0.0166 = 1.66\% \]

(6) \[ P(0.52 \leq Z \leq 1.94) \]

\[ = P(0 \leq Z \leq 1.94) - P(0 \leq Z \leq 0.52) \]

\[ = 0.4756 - 0.1985 - 0.2753 = 27.53\% \]

\[ P(1.07 \leq Z \leq 2.59) \]

\[ = P(0 \leq Z \leq 2.59) - P(0 \leq Z \leq 1.07) \]

\[ = 0.4952 - 0.3577 - 0.1375 = 13.75\% \]
The Area under the Standard Normal Distribution

(7) \( P(-1.33 \leq Z \leq 0) = P(0 \leq Z \leq 1.33) = 0.4082 \)

(8) \( P(Z \leq -1.47) = P(Z > 1.47) = 0.5 - P(0 \leq Z \leq 1.47) \)

\[ = 0.5 - 0.4292 = 0.0708 = 7.08\% \]

The Area under the Standard Normal Distribution

(9) \( P(-1.83 \leq Z \leq -0.85) = P(0.85 \leq Z \leq 1.83) \)

\[ = P(0 \leq Z \leq 1.83) - P(0 \leq Z \leq 0.85) \]

\[ = 0.4664 - 0.3023 - 0.1641 = 16.41\% \]

(10) \( P(-1.23 \leq Z \leq 1.75) = P(0 \leq Z \leq 1.75) + P(-1.23 \leq Z \leq 0) \)

\[ = P(0 \leq Z \leq 1.75) + P(0 \leq Z \leq 1.23) \]

\[ = 0.4599 + 0.3907 - 0.8506 = 85.66\% \]
The Area under the Standard Normal Distribution

(11) \[ P(1.5 \leq Z \leq 1.5) = 2 \cdot P(0 \leq Z \leq 1.5) = 2 \cdot 0.4332 = 0.8664 \]

(12) \[ P(Z \geq 1.2) = P(Z \leq -1.2) + P(Z \geq 1.2) = 2 \cdot [0.5 - P(0 \leq Z \leq 1.2)] = 2 \cdot [0.5 - 0.3849] = 2 \cdot 0.1151 = 0.2302 \]

The Area under Any Normal Distribution

If X is a R.V. that follows the normal distribution with mean (\( \mu \)) equals 100 and Standard Deviation (\( \sigma \)) equals 25. We write this in a symbolic form as:

\[ X \sim N(\mu = 100, \sigma = 25) \quad \text{OR simple as} \quad X \sim N(100, 25) \]

Find the probability that: \( 80 < X < 130 \)
The Area under Any Normal Distribution

To find the probability we will use the standardizing formula, which will find the equivalent area under the standard Normal Distribution. This formula is:

\[
\text{Standard value of } X = \frac{x - \text{mean of } (X)}{\text{St. Dev. of } (X)}
\]

<table>
<thead>
<tr>
<th>Standard Value of X</th>
<th>is</th>
<th>( \frac{100-100}{25} )</th>
<th>=</th>
<th>( \frac{80-100}{25} )</th>
<th>=</th>
<th>( \frac{130-100}{25} )</th>
<th>=</th>
</tr>
</thead>
<tbody>
<tr>
<td>= 0</td>
<td>25</td>
<td>0</td>
<td>0.8</td>
<td>-0.8</td>
<td>1.2</td>
<td></td>
<td>1.2</td>
</tr>
</tbody>
</table>

\[
P(80 \leq X \leq 130) = P\left(\frac{80-100}{25} \leq \frac{x - \mu}{\sigma} \leq \frac{130-100}{25}\right)
\]

\[
= P[-0.8 \leq Z \leq 1.2]
\]

\[
P(0 \leq Z \leq 1.2) + P(0 \leq Z \leq 0.8)
\]
Determining the “z” value when the area under the standard Normal Distribution is known

- If we know the area under the Standard Normal Distribution from 0 to a positive value Z. We then can use the standard normal table to find the value “z”.
- this is done in a reverse process to what we have done previous, when we knew “z” and we needed to find out the area.
- To determine the value of “z” we have to look at the value of the probability in the table that is equal to the area, or the closest value to it (closest but not larger than it).
- The value of “z” is determined as the interaction of the row and column of the cell that contains the closest value to the specified area of concern.

<table>
<thead>
<tr>
<th>Z</th>
<th>0.00</th>
<th>0.01</th>
<th>0.02</th>
<th>0.03</th>
<th>0.04</th>
<th>0.05</th>
<th>0.06</th>
<th>0.07</th>
<th>0.08</th>
<th>0.09</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>0.0550</td>
<td>0.0580</td>
<td>0.0610</td>
<td>0.0640</td>
<td>0.0670</td>
<td>0.0700</td>
<td>0.0730</td>
<td>0.0760</td>
<td>0.0790</td>
<td>0.0820</td>
</tr>
<tr>
<td>0.2</td>
<td>0.1150</td>
<td>0.1190</td>
<td>0.1230</td>
<td>0.1270</td>
<td>0.1310</td>
<td>0.1350</td>
<td>0.1390</td>
<td>0.1430</td>
<td>0.1470</td>
<td>0.1510</td>
</tr>
<tr>
<td>0.3</td>
<td>0.1860</td>
<td>0.1900</td>
<td>0.1940</td>
<td>0.1980</td>
<td>0.2020</td>
<td>0.2060</td>
<td>0.2100</td>
<td>0.2140</td>
<td>0.2180</td>
<td>0.2220</td>
</tr>
<tr>
<td>0.4</td>
<td>0.2440</td>
<td>0.2480</td>
<td>0.2520</td>
<td>0.2560</td>
<td>0.2600</td>
<td>0.2640</td>
<td>0.2680</td>
<td>0.2720</td>
<td>0.2760</td>
<td>0.2800</td>
</tr>
<tr>
<td>0.5</td>
<td>0.3290</td>
<td>0.3330</td>
<td>0.3360</td>
<td>0.3400</td>
<td>0.3430</td>
<td>0.3460</td>
<td>0.3490</td>
<td>0.3520</td>
<td>0.3550</td>
<td>0.3580</td>
</tr>
</tbody>
</table>

Area Under Standard Normal Distribution P(0 < Z < z)
Determining the “z” value when the area under the standard Normal Distribution is known

Example -1: If we know that the area under the standard normal curve from \( Z=0 \) to a positive value \( Z \) equals 39.5%. Find the value “z” that would give such an area?

What is the value of "z" such that: \( P(0 \leq Z \leq z) = 0.3950 \)

- look inside the table of the closest value to 0.3950
- We will not find a value in the table exactly equal to 0.3950
- The closest values are: 0.3944 and the next value to that is 0.3962
- So, we will take 0.3944 as the one closest to 0.3950
- The value of \( z \) that gives this area is read from the row and column of that cell as \( z = 1.25 \)

Example -2: Find the value “z” such that: \( P(-z \leq Z \leq z) = 0.8400 \)

Here the area is to the left and right of the 0 value
- The table gives the area to the right side of the 0

- So, we will find the value of "z" such that the area from 0 to positive "z" equals half of the given area, i.e. \( 0.8400/2 = 0.4200 \)
- the closest value to the area is : 0.4192 and the value of "z" for that is 1.40
- So, \( P(-1.40 \leq Z \leq 1.40) = 0.8400 \)
Determining the “z” value when the area under the standard Normal Distribution is known

Example -3: From the previous example we can conclude the following:

- The value of Z such that $P(-z \leq Z \leq z) = 0.90$ is $z = 1.64$
  i.e. $P(-1.64 \leq Z \leq 1.64) = 0.90$

- The value of Z such that $P(-z \leq Z \leq z) = 0.95$ is $z = 1.96$
  i.e. $P(-1.96 \leq Z \leq 1.96) = 0.95$

- The value of Z such that $P(-z \leq Z \leq z) = 0.99$ is $z = 2.57$
  i.e. $P(-2.57 \leq Z \leq 2.57) = 0.99$

Determining the “x” value when the area under any Normal Distribution is known

- If we know the area under the Normal Distribution from the mean ($\mu$) to a value $X$ greater than $\mu$

- We then can use the transformation to the standard normal distribution and find the equivalent area under the standard normal distribution

- Then use the standard normal table to find the value “z”

- From the value of “z” we use the standard normal transformation formula again to find out the value of $X$
Example -1: If we know that $X$ is a random variable that follows the normal distribution with mean equals to 200 and standard deviation 50. What is the value of “$x$” such that $P(200 < X < x) = 0.4600$?

Find the equivalent area under the standard normal distribution (i.e. the value “$z$”) that would give such area.

We can write the above problem as:

$P(200 \leq X \leq x) = 0.4600$

$= P\left(\frac{200 - 200}{50} \leq \frac{X - \mu}{\sigma} \leq \frac{x - 200}{50}\right)$

$= P\left(0 \leq Z \leq z\right) = 0.4600$

Determining the “$x$” value when the area under any Normal Distribution is known

Example -1: cont.

From the standard normal table $z = 1.75$. Solving for $x$ we have:

$z = 1.75 = \frac{x - 200}{50}$

$50z = x - 200$

$x = 50*1.75 + 200 = 287.5$
Determining the “x” value when the area under any Normal Distribution is known

Example -2: If X is a random variable with mean equals 120 and standard deviation equals 25. Find the value “x” such that:  
\[ P(-\infty \leq X \leq x) = 0.93 \]

this is equivalent to:
\[ P(X \leq x) = 0.93 \]

This means that:
\[ P(120 \leq X \leq x) = 0.93 - 0.5 = 0.43 \]

Example -2: cont.
To find x we have to find the equivalent area under the standard normal curve
\[
P(120 \leq X \leq x) = 0.43
\]
\[
= P\left(\frac{120-120}{25} \leq \frac{X-\mu}{\sigma} \leq \frac{x-120}{25}\right)
\]
\[
= P\left(0 \leq Z \leq z\right) = 0.43
\]

From the standard normal table z = 1.47. From that we can solve for x as following:
\[
z = 1.47 = \frac{x-120}{25}
\]

which implies that:
\[
x = (25 \times 1.47) + 120 = 156.75
\]