Lecture Notes on "Pre-Calculus for Business (Math 98)"

Chapter 1:
Applications of Equations and Inequalities

Text Book: "Introductory Mathematical Analysis"
Instructor: Mohammad Reda Almohri
Chapter 1: Applications of Equations and Inequalities

Note: The solutions of the stared problems are provided at the end of the chapter.

1.1 Applications of equations:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(c)</td>
<td>cost</td>
</tr>
<tr>
<td>(q)</td>
<td>quantity [number of units]</td>
</tr>
<tr>
<td>(p)</td>
<td>price</td>
</tr>
<tr>
<td>(r)</td>
<td>revenue</td>
</tr>
<tr>
<td>(P)</td>
<td>Profit</td>
</tr>
</tbody>
</table>

\[
\text{Total cost} = \text{Variable cost} + \text{Fixed cost}
\]

\[
\text{Variable cost} = (\text{cost per unit}) \times (\text{quantity})
\]

\[
\text{Total revenue} = \text{price per unit} \times \text{quantity}
\]

\[
\text{Profit} = \text{Total revenue} - \text{Total cost}
\]

**EXAMPLE:** (1) (Profit)

The variable cost per unit for a product is 7KD and fixed cost is 87,000KD. Each unit has a selling price of 13KD. Determine the number of units that must be sold for the company to earn a profit of 72,000KD.

**SOLUTION:**

\[
\text{cost per unit} = 7KD
\]

\[
\therefore \text{Variable cost} = 7q
\]

Fixed cost = 87,000KD

Total cost = Variable cost + Fixed cost = 7q + 87,000

Price per unit = 13KD

profit
\[ \text{Total revenue} = \text{(price per unit)} \times \text{(quantity)} = 13q \]

Profit = 72,000KD

Profit = Total revenue – Total cost

\[
\begin{align*}
\therefore 72,000 &= 13q - (7q + 87,000) \\
13q - 7q &= 72,000 + 87,000 \\
6q &= 159,000 \\
q &= 159,000/6 = 26,500
\end{align*}
\]

**EXAMPLE:** (2) (Profit)  

The variable cost per unit of a product is 1.700KD. If fixed costs are 50,000KD and each unit sells for 2.200KD, how many units must be sold to have a profit of 35,000KD?

\[
\begin{align*}
\text{Variable cost} &= 1.700q \\
\text{Total cost} &= \text{Variable cost} + \text{Fixed cost} \\
&= 1.700q + 50,000 \\
\text{Total revenue} &= 2.200q \\
\text{Profit} &= \text{Total revenue} - \text{Total cost} \\
&= 2.200q - (1.700q + 50,000) \\
&= 2.2q - 1.7q - 50,000 = 0.5q - 50,000 \\
35,000 + 50,000 &= 0.5q \\
85,000 &= 0.5q \\
q &= \frac{85,000}{0.5} = 85,000 \times 2 = 170,000
\end{align*}
\]

**EXAMPLE:** (3) (Profit)  

The following data are available for a company: unit selling price of 14KD; variable cost per unit of 11.750KD; fixed cost of 78,925KD. From these data determine the total sales units that are required for the company to earn a profit of 50,000KD.

\[
\begin{align*}
\text{Variable cost} &= (11.750)q \\
\text{Total cost} &= \text{Variable cost} + \text{Fixed cost} \\
&= (11.750)q + 78,925 \\
\text{Total revenue} &= 14q \\
\text{Profit} &= \text{Total revenue} - \text{Total cost} \\
50,000 &= 14q - [(11.750)q + 78,925] \\
&= 14q - (11.750)q - 78,925 = (2.250)q - 78,925
\end{align*}
\]
EXAMPLE: \((\boldsymbol{4})\) \(\text{(investment)}\) \(\text{[EXAMPLE5]}\)

A total of 12,000KD was invested in two business ventures, \(A\) and \(B\). At the end of the first year, \(A\) and \(B\) yielded returns of 6.5% and 5.5%, respectively, on the original investments. How was the original amount allocated if the total amount earned was 710KD?

\[
\begin{align*}
50,000 + 78,925 &= (2.250)q \\
128,925 &= (2.250)q \\
q &= \frac{128,925}{(2.250)} = 57,300
\end{align*}
\]

EXAMPLE: \((\boldsymbol{5})\) \(\text{(investment)}\) \(\text{[EXERCISE 11]}\)

A person wishes to invest 7,500KD in two enterprises so that the total income per year will be 510KD. One enterprise pays 6.5% annually; the other pays 7% annually. How much must be invested in each?

\[
\begin{align*}
50,000 + 78,925 &= (2.250)q \\
128,925 &= (2.250)q \\
q &= \frac{128,925}{(2.250)} = 57,300
\end{align*}
\]

\[
\begin{align*}
\text{Suppose that:} \\
\text{The amount invested in 6.5\%} &= x \quad \text{(Investment A)}
\end{align*}
\]
\[ \therefore \text{The amount invested in } 7\% = 7,500 - x \quad \text{(Investment B)} \]

(Interest in A) \[= \frac{6.5}{100} \cdot x = 0.065x \]

(Interest in B) \[= \frac{7}{100} \cdot (7,500 - x) = \frac{7}{100} \cdot 7,500 - \frac{7}{100} \cdot x \]
[\[= 7 \times 75 - 0.07x = 525 - 0.07x \]

Total interest = interest in A + interest in B
\[510 = 0.065x + 525 - 0.07x \]
\[510 = 525 - 0.005x \]
\[0.005x = 525 - 510 = 15 \]
\[\therefore x = \frac{15}{0.005} = \frac{15,000}{5} = 3,000 \]

(Investment A) = \[x = 3000K D\]
(Investment B) = \[7,500 - x = 7,500 - 3,000 = 4,500K D\]

**EXAMPLE:** (٦) (Investment) \[\text{[EXERCISE 12]}\]

A person invested 2,000KD, part at an interest rate of 3% annually and the remainder at 5% annually. The total interest at the end of one year was equivalent to an annual \(4\frac{1}{4}\%\) rate on the entire 2,000KD. How much was invested at each rate?

Suppose that:

The amount invested in 3% = \(x\) \quad \text{(Investment A)}
\[\therefore \text{The amount invested in } 5\% = 2,000 - x \quad \text{(Investment B)} \]

(Interest in A) \[= \frac{3}{100} \cdot x = 0.03x \]

(Interest in B) \[= \frac{5}{100} \cdot (2,000 - x) = \frac{5}{100} \cdot 2,000 - \frac{5}{100} \cdot x \]
[\[= 100 - 0.05x \]

The total interest at the end of one year = \(\frac{4\frac{1}{4}}{100} \cdot (2,000)\)
[\[= \frac{4.25}{100} \cdot (2,000) = 4.25 \times 20 = 85 \]

Total interest = interest in A + interest in B
\[85 = 0.03x + 100 - 0.05x \]
\[= 100 - 0.02x \]
\[0.02x = 100 - 85 = 15 \]
\[\therefore x = \frac{15}{0.02} = 750 \]

(Investment A) = \(x = 750K D\)

SOLUTION:

استثمر شخص ٢٠٠٠ دينار في مشروعين ليحصل على ٣% فائدة سنوية من الأول و٥% من الثاني. كانت الفائدة الكلية في نهاية سنة واحدة تعادل \(4\frac{1}{4}\%\) من الاستثمار الأصلي ٢٠٠٠ دينار. كم استثمر في كل منهما؟

<table>
<thead>
<tr>
<th>Solution</th>
<th>Investment A</th>
<th>Investment B</th>
</tr>
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<tbody>
<tr>
<td>Invested</td>
<td>3000 KD</td>
<td>4500 KD</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Pastimes</th>
<th>Exercise 12</th>
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<tr>
<td>A person invested 2,000 KD, part at an interest rate of 3% annually and the remainder at 5% annually. The total interest at the end of one year was equivalent to an annual (4\frac{1}{4}%) rate on the entire 2,000 KD. How much was invested at each rate?</td>
<td>Suppose that: The amount invested in 3% = (x) (Investment A) The amount invested in 5% = 2,000 - (x) (Investment B)</td>
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</table>
EXAMPLE: (v) (Bond Redemption [سامد الدين]) [EXAMPLE 6]

A firm decided to redeem 676,000KD of its bonds in two years. The firm presently sets aside 625,000KD. At what annual rate of interest, compounded annually, will this money have to be invested in order that its value be sufficient to redeem the bonds?

SOLUTION:
Let \( r \) be the required annual rate of interest. At the end of the first year the amount will be:

\[
625,000 + 625,000r = 625,000(1+r)
\]

At the end of the second year the amount will be:

\[
[625,000(1+r)] + [625,000(1+r)]r = [625,000(1+r)](1+r) = 625,000(1+r)^2
\]

This must equal 676,000KD:

\[
(1+r)^2 = \frac{676,000}{625,000} = \frac{676}{625} = \left(\frac{26}{25}\right)^2 = \left(\frac{26}{25}\right)^2
\]

\[
1 + r = \pm \sqrt{\frac{26}{25}} = \pm \frac{26}{25} = \pm 1.04
\]

\[
r = \pm 1.04 - 1
\]

\[
\therefore r = 1.04 - 1 = 0.04 < 0 \quad \text{(unacceptable)}
\]

or \( r = 0.04 - 1 = 0.04 \)

The annual rate of interest = 4%

EXAMPLE: (viii) (Bond Retirement [العهد]) [EXERCISE 14]

In three years a company will require $2,315,250KD in order to retire some bonds. If the company now invests 2,000,000KD for this purpose, what annual rate of interest, compounded annually, must it receive on that amount in order to retire the bonds?

SOLUTION:
Let \( r \) be the required annual rate of interest. At the end of the first year, the amount will be:

\[
2,000,000 + 2,000,000r = 2,000,000(1+r)
\]

At the end of the second year the amount will be:

\[
2,000,000(1+r)^2
\]
At the end of the third year the amount will be:

\[
2,000,000(1+r)^3 + 2,000,000(1+r)^2 \cdot r = 2,000,000(1+r)^3(1+r) = 2,000,000(1+r)^3
\]

This must equal 2,315,250KD: 2,000,000(1+r)^3 = 2,315,250

\[
(1+r)^3 = \frac{2,315,250}{2,000,000} = \frac{1,157,625}{1,000,000}
\]

\[
1 + r = \sqrt[3]{\frac{1,157,625}{1,000,000}} = \frac{105}{100} = 1.05
\]

\[
r = 1.05 - 1 = 0.05
\]

The annual rate of interest = 5%.

**Profit = selling price − cost**

**Selling price = tag price − reduction**

**EXAMPLE: (9) (Pricing)**

The cost per unit is 48KD. A price tag will be attached to each product. What amount should be marked on the price tag so that it may be reduced by 20% during a sale and still make a profit of 15% on the cost?

SOLUTION:

For a unit:

- Cost = 48KD
- Profit = 15% on the cost = \( \frac{15}{100} \times 48 = 0.15 \times 48 = 7.2KD \)
- Profit = selling price − cost
- 7.2 = selling price − 48
- ∴ Selling price = 7.2 + 48 = 55.2KD
- Suppose that: tag price = \( p \)
- ∴ Reduction = \( \frac{20}{100} p = 0.2 p \)
- Selling price = tag price − reduction
- ∴ 55.2 = \( p - 0.2 p \)
- 55.2 = 0.8 \( p \)
- \( p = \frac{55.2}{0.8} = \frac{552}{8} = 69KD \)
EXAMPLE: (10) (pricing) [EXERCISE 13]

The cost of a product to a retailer is 161KD. If the retailer wishes to make a profit of 30% on the selling price, at what price should the product be sold?

If an item costs 161 dinars and the retailer wishes to make a 30% profit, at what selling price should the product be sold?

SOLUTION:
Let \( x \) be the required selling price, \( P \) the profit, and \( c \) the cost.

\[
\therefore P = \frac{30}{100} x = 0.3x .
\]

But \( P = x - c \)
\[
\therefore 0.3x = x - 161 \\
0.3x - x = -161 \Rightarrow -0.7x = -161 \Rightarrow x = \frac{-161}{-0.7} = \frac{1610}{7} = 230KD
\]

EXAMPLE: (11) (expansion program) [EXERCISE 15]

A company will begin an expansion program. It has decided to invest 100,000KD now so that in one year the total value of the investment will be 105,750KD, the amount required for the expansion. What is the annual rate of interest, compounded annually, that the company must receive to achieve its purpose?

Let \( r \) be the required annual rate of interest. At the end one year the amount will be:

\[
100,000 + 100,000r = 100,000(1 + r)
\]

This must equal 105,750KD: \( 100,000(1 + r) = 105,750 \)
\[
1 + r = \frac{105,750}{100,000} = 1.0575r = 1.0575 - 1 = 0.0575 \\
\therefore r = 0.0575 = \frac{5.75}{10,000} = \frac{5.75}{100} = 5.75\% = \frac{5.75}{4}\%
\]

The annual rate of interest = 5.75%.

EXAMPLE: (12) (Business) [EXERCISE 16]

A company finds that if it produces and sells \( q \) units of a product, its total sales revenue in dinars is \( 45\sqrt{q} \). If the variable cost per unit is 3KD and the fixed cost is 150KD, find the values of \( q \) for which

\[
\text{total revenue} = \text{total cost} \quad \text{(That is profit is zero.)}
\]

WANTED: \( q \) يبين كمية المنتج الكلي لكل وحدة $45\sqrt{q}$، وبعدها احتساب الربح.
SOLUTION:

Total cost = variable cost + fixed cost = 3q + 150
Total revenue = total cost

\[
45\sqrt{q} = 3q + 150
\]

\[
\left(45\sqrt{q}\right)^2 = (3q + 150)^2 \Rightarrow 2025q = 9q^2 + 900q + 22,500
\]

\[9q^2 + 900q + 22,500 - 2025q = 0\]

\[9q^2 - 1125q + 22,500 = 0 \Rightarrow q^2 - 125q + 2500 = 0\]

\[q - 25)(q - 100) = 0 \Rightarrow q = 25 \text{ or } q = 100\]

EXAMPLE: (13) (revenue) [EXERCISE 25]

Suppose that consumers will purchase \( q \) units of a product when the price is \( 14 - \left(\frac{q}{5}\right) \) dinars each. How many units must be sold in order that sales revenue be 240KD?

\[
p = 14 - \left(\frac{q}{5}\right)
\]

\[
r = pq \Rightarrow 240 = q \left[14 - \left(\frac{q}{5}\right)\right] = 14q - \left(\frac{q^2}{5}\right)
\]

\[240 = 14q - \frac{q^2}{5} \Rightarrow 1200 = 70q - q^2 \Rightarrow q^2 - 70q + 1200 = 0
\]

\[
\therefore (q - 30)(q - 40) = 0 \Rightarrow q = 30 \text{ or } q = 40
\]

EXAMPLE: (14) (Rentals) [EXERCISE 29]

An office complex with 40 units must be rented. At 360KD per month, every unit can be rented. However, for each 15KD per month increase, there will be one unit with no possibility of filling it. Determine the rent that should be charged for each unit in order to receive a total of 15,300KD per month?

SOLUTION:

Suppose that the rent is increased of \( 15x \) KD per month. Hence, only \( 40 - x \) units will be rented and the rent of each is \( 360 + 15x \).

\[
\therefore (40 - x)(360 + 15x) = 15,300
\]

\[14,400 + 240x - 15x^2 = 15,300\]

\[15x^2 - 240x - 14,400 + 15,300 = 0 \Rightarrow 15x^2 - 240x + 900 = 0\]

\[x^2 - 16x + 60 = 0 \Rightarrow (x - 10)(x - 6) = 0 \Rightarrow x = 10 \text{ or } x = 6\]
The rent for each mune be: $360 + (15 \times 10) = 510KD$

or $360 + (15 \times 6) = 450KD$.

1.2 Linear Inequalities:

For any two points on the real-numbers line $a$ and $b$ we have one of the three cases:

1. $a$ and $b$ coincide: which means that $a=b$.

2. $a$ lies to the left of $b$: then $a<b$ ($a$ is less than $b$) or $b>a$ ($b$ is greater than $a$).

3. $a$ lies to the right of $b$: then $a>b$ ($a$ is greater than $b$) or $b<a$ ($b$ is less than $a$).

Inequality symbols:

1. $<$: less than
2. $>$: greater than
3. $\leq$: less than or equal
4. $\geq$: greater than or equal

Notes:

1. $a > 0$: $a$ is positive (+ve)
2. $a < 0$: $a$ is negative (−ve)
3. $a < x < b$: $a < x$ and $x < b$

Similarly for other symbols: $>$, $\leq$ and $\geq$. For example $a \geq x > b$ means $a \geq x$ and $x > b$.

Solving an Inequality: Finding all values of the variable for which the inequality is true.

Rules for Inequalities:

For any real numbers $a, b$ and $c$:

1. If $a < b$ then: $a + c < b + c$, and $a - c < b - c$.
Similarly for the other symbols: $>, \leq$ and $\geq$.

**EXAMPLE:** (1) $6 < 10 \Rightarrow 6 + 3 < 10 + 3 \Rightarrow 9 < 13$

(2) $9 \geq 2 \Rightarrow 9 - 4 \geq 2 - 4 \Rightarrow 5 \geq -2$

2. If $a < b$ and $c$ is positive then: $ac < bc$ , and $\frac{a}{c} < \frac{b}{c}$.

Similarly for the other symbols: $>, \leq$ and $\geq$.

**EXAMPLE:** (1) $2 < 5 \Rightarrow 2 \times 3 < 5 \times 3 \Rightarrow 6 < 15$

(2) $1 > -3 \Rightarrow 1(4) > -3(4) \Rightarrow 4 > -12$

(3) $-2 \geq -7 \Rightarrow -2(3) \geq -7(3) \Rightarrow -6 \geq -21$

3. If $a < b$ and $c$ is negative then: $ac > bc$ , and $\frac{a}{c} > \frac{b}{c}$.

Similarly for the other symbols: $>, \leq$ and $\geq$. For example: If $a \geq b$ and $c$ is negative then $ac \leq bc$ , and $\frac{a}{c} \leq \frac{b}{c}$.

**EXAMPLE:** (1) $5 > 2 \Rightarrow 5(-1) < 2(-1) \Rightarrow -5 < -2$

(2) $-4 \leq 5 \Rightarrow -4(-2) \geq 5(-2) \Rightarrow 8 \geq -10$

(3) $-1 < 0 \Rightarrow -1(-5) > 0(-5) \Rightarrow 5 > 0$

4. If $a < b$ and $a = c$ then $c < b$.

Similarly for the other symbols: $>, \leq$ and $\geq$.

**EXAMPLE:** (1) If $x < 2$ and $y = x$ then $y < 2$

(2) If $3 \geq t$ and $t = 2x$ then $3 \geq 2x$.

5. If $0 < a < b$ then $\frac{1}{a} > \frac{1}{b}$ (0 < a < b means that a and b are both positive).

If $a < b < 0$ then $\frac{1}{a} > \frac{1}{b}$ (a < b < 0 means that a and b are both negative).

i.e., if a and b are both positive or both negative and $a < b$ then $\frac{1}{a} > \frac{1}{b}$.
\[
\frac{1}{a} > \frac{1}{b} \quad \text{if} \quad a < b
\]

EXAMPLE: (1) \( 2 < 3 \) \( \Rightarrow \) \( \frac{1}{2} > \frac{1}{3} \)

(2) \(-4 > -9 \) \( \Rightarrow \) \( \frac{1}{-4} < \frac{1}{-9} \)

(3) \( \frac{3}{2} \leq \frac{7}{4} \) \( \Rightarrow \) \( \frac{2}{3} \geq \frac{4}{7} \)

(4) \( -2 \leq -\frac{1}{5} \) \( \Rightarrow \) \( -\frac{1}{2} \geq -5 \)

Note: Rule 5 is not true if \( a \) and \( b \) have different symbols, for example: \( -2 < 3 \) and \( -\frac{1}{2} < -\frac{1}{3} \).

6. If \( 0 < a < b \) and \( n \) is a positive integer then: \( a^n < b^n \) and \( \sqrt[n]{a} < \sqrt[n]{b} \).

Example: (1) \( 2 < 3 \) \( \Rightarrow \) \( 2^4 = 16 < 3^4 = 81 \)

(2) \( 27 > 8 \) \( \Rightarrow \) \( \sqrt[3]{27} = 3 > \sqrt[3]{8} = 2 \)

(3) \( 64 < 100 \) \( \Rightarrow \) \( (64)^{1/2} = \sqrt{64} = 8 < (100)^{1/2} = \sqrt{100} = 10 \)

Equivalent Inequalities: are equalities that have the same solutions. Rules 1 – 4 results equivalent inequalities.

Intervals: For any two real numbers \( a \) and \( b \) such that \( a < b \), we define:

1. \( (a, b) = \{ x : a < x < b \} \)

\( (a, b) \) is called an open interval.

2. \( [a, b) = \{ x : a \leq x < b \} \)

3. \( (a, b] = \{ x : a < x \leq b \} \)

4. \( [a, b] = \{ x : a \leq x \leq b \} \)

\( [a, b] \) is called a closed interval.
5. \((a, \infty) = \{x : a < x\}\)

The symbol \(\infty\) is called infinity.

6. \([a, \infty) = \{x : a \leq x\}\)

7. \((-\infty, a) = \{x : x < a\}\)

8. \((-\infty, a] = \{x : x \leq a\}\)

9. \((-\infty, \infty) = \mathbb{R}\), all real numbers.

**Linear Inequality:**

A linear inequality is an inequality that can be written in the forms:

\[
ax + b < 0, \quad ax + b \leq 0, \quad ax + b > 0, \quad \text{or} \quad ax + b \geq 0,
\]

where \(a\) and \(b\) are constants and \(a \neq 0\).

**EXAMPLE:** Solve the following inequalities:

1. \(5x + 8 < 3\)
   
   **SOLUTION:**
   
   \(5x < 3 - 8\) \(\Rightarrow\) \(5x < -5\) \(\Rightarrow\) \(x < -1\)
   
   \(\therefore\) Solution set = \((-\infty, -1)\)

2. \(\frac{x + 1}{3} - 2 \leq \frac{x}{2} + 1\)
   
   **SOLUTION:**
   
   \(2(x + 1) - 12 \leq 3x + 6\) \((\text{Multiplying both sides by 6})\)
   
   \(2x + 2 - 12 \leq 3x + 6\) \(\Rightarrow\) \(2x - 3x \leq 6 - 2 + 12\) \(\Rightarrow\) \(-x \leq 16\)
   
   \(\therefore\) \(x \geq -16\) \((\text{Multiplying both sides by}-1\text{ and changing the symbol because }-1 < 0)\)
   
   \(\therefore\) Solution set = \([-16, \infty)\)

3. \(4 - (2x + 1) \geq -2(2) - 5\)
   
   **SOLUTION:**
   
   \(4 - 2x - 1 \geq -2x + 10\) \(\Rightarrow\) \(-2x + 2x \geq 10 - 4 + 1\) \(\Rightarrow\) \(0 \geq 7\) \((\text{impossible})\)
   
   \(\therefore\) Solution set = \(\emptyset\)

4. \(5x - 3(1 - x) < 2(4x + 3)\)
   
   **SOLUTION:**
   
   \(5x - 3+ 3x < 8x + 6\) \(\Rightarrow\) \(8x - 8x < 6 + 3\) \(\Rightarrow\) \(0 < 9\) \((\text{always true})\)
   
   \(\therefore\) Solution set = \(\emptyset\)
EXERCISES 1.2:

In problems 1 – 20, solve the given inequalities:

1. \(3x + 7 < 4\)
2. \(1 - 5y \geq -2\)
3. \(100y < 350y\)
4. \(\frac{2}{7}t + 1 < 2 - \frac{3}{2}t\)
5. \(\sqrt{3}(w - 2) > \frac{w}{\sqrt{2}}\)
6. \(2 - \frac{x - 5}{4} \leq \frac{1}{3} - 2x\)
7. \(\frac{4x - 8x^2}{2} < \frac{3x^2 - 1}{3} - 5x^2 + 4x\)
8. \(4\left(3p^2 - p\right) \geq 17 + 3\left(1 - 2p\right)^2\)
9. \(2(4y + 7) - 5y \leq 3 - 3(6 - 2y)\)
10. \(2x(5 - 2x) + 4 \leq 13 - 4(x - 1)^2 + 2x\)
11. \(4(x - 7) + 5x > 3(3x - 1) + 2\)
12. \(-7(x + 2) \leq -x\)
* 13. \(2(4 - x) - 1 > 3(x + 3) - 5(2 - x)\)
* 14. \(\frac{3x^2 - 7x}{3} < x^2 + \frac{3x - 1}{2}\)
* 15. \(\frac{y + 1}{6} \leq -\frac{3}{2} - \frac{y}{3}\)
* 16. \(\frac{3}{2}x + 2 < \frac{7}{6}x - 1\)
* 17. \(4s - 1 \geq -2(3 - 2s) + 5\)
* 18. \(3x - 7 \geq x - 2(3 - x)\)
* 19. \(2(-3t - 5) + 11 > 5 - 2(6 + 2t)\)
* 20. \((1 - 3x)^2 - 9(2 - x)^2 \geq 5\)

* 21. Suppose that \(a < 0\) and \(b > 0\). Classify the following expressions as either true or false:

| a. \(ab > 0\) | b. \(a \leq b\) | c. \(b^2 \geq b\) |
| d. \(a/b < 0\) | e. \(a - b < 0\) | f. \(a^2 > a\) |

1.4 Absolute Value:

**Definition:** \(|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}\)

\(|x|\) is the distance of \(x\) from 0.

**EXAMPLE:**

\(-6 -5 -4 -3 -2 -1 0 1 2 3 4 5\)

Solving Absolute-Value Equations:

**EXAMPLE:** Solve the following equations:
(1) \(|x - 5| = 3\)

**SOLUTION:** \(|x - 5| = 3\) means \(x - 5 = 3\) or \(x - 5 = -3\)

\[x = 8, x = 2\]

\[
\therefore \text{ Solution set } = \{2, 8\}
\]

(2) \(|4x + 3| = 7\)

**SOLUTION:**

Case 1: \(4x + 3 = 7\) \(\Rightarrow x = 1\)

Case 2: \(4x + 3 = -7\) \(\Rightarrow x = -5/2\)

\[
\therefore \text{ Solution set } = \{1, -5/2\}
\]

(3) \(|4 - 3x| = -2\)

**SOLUTION:** Solution set = \(\emptyset\), because the absolute value will never be negative.

The distance between two points:

For any two real numbers \(a\) and \(b\), we have:

1. \(|b - a| = |a - b|\)

2. \(|b - a|\) is the distance between \(a\) and \(b\).

**Example:** \(|2 - 7| = |7 - 2| = 5\)

Absolute-value inequalities:

If \(d > 0\), then for any \(y\):

1. \(|y| < d \iff -d < y < d\)

2. \(|y| \leq d \iff -d \leq y \leq d\)

3. \(|y| > d \iff y > d\ or \ y < -d\)

4. \(|y| \geq d \iff y \geq d\ or \ y \leq -d\)

Solving Absolute-value inequalities:

**Example:** Solve the following inequalities:
(1) \[ |x - 5| < 2 \]
**SOLUTION:** 
\[ |x - 5| < 2 \implies -2 < x - 5 < 2 \implies 5 - 2 < x < 5 + 2 \implies 3 < x < 7 \]
\[ \therefore \text{ Solution set } = (3, 7) \]

(2) \[ |7 - 3x| \leq 1 \]
**SOLUTION:** 
\[ |7 - 3x| \leq 1 \implies -1 \leq 7 - 3x \leq 1 \implies -7 - 1 \leq -3x \leq -7 + 1 \]
\[ -8 \leq -3x \leq -6 \implies \frac{-8}{-3} \geq x \geq \frac{-6}{-3} \implies \frac{8}{3} \geq x \geq 2 \implies 2 \leq x \leq \frac{8}{3} \]
\[ \therefore \text{ Solution set } = [2, \frac{8}{3}] \]

(3) \[ |2x + 1| \geq 6 \]
**SOLUTION:** Case 1: \[ 2x + 1 \geq 6 \implies 2x \geq 5 \implies x \geq \frac{5}{2} \]
Case 2: \[ 2x + 1 \leq -6 \implies 2x \leq -7 \implies x \leq -\frac{7}{2} \]
\[ \therefore \text{ Solution set } = (-\infty, -\frac{7}{2}] \cup [\frac{5}{2}, \infty) \]

**Note:** \( \cup \) means the union of two sets. Suppose that \( A \) and \( B \) are two sets. We define:
\[ A \cup B = \{ x : x \text{ belongs to } A \text{ or } x \text{ belongs to } B \} \]
\( A \cup B \) is a set gathers all elements of \( A \) and \( B \).

(4) \[ |2 - 3x| > 3 \]
**SOLUTION:** Case 1: \[ 2 - 3x > 3 \implies -3x > 1 \implies x < -\frac{1}{3} \]
Case 2: \[ 2 - 3x < -3 \implies -3x < -5 \implies x > \frac{-5}{-3} \implies x > \frac{5}{3} \]
\[ \therefore \text{ Solution set } (-\infty, -\frac{1}{3}) \cup (\frac{5}{3}, \infty) \]

**Absolute-Value Notation:**

**EXAMPLE:** Using absolute-value notation, express the following statements:

(1) \( x \) is less than 7 units from 2.
**SOLUTION:** \( |x - 2| < 7 \)

(2) \( y \) is more than 1 units from -9.
**SOLUTION:** \( |y - (-9)| > 1 \)

**Note:** \( |x| \) denotes the absolute value of \( x \), which is the non-negative value of \( x \) itself without regard to its sign.
SOLUTION: \(|y - (-9)| > 1\) or \(|y + 9| > 1\)

(3) \(t\) differs from 8 by at least 3.

SOLUTION: \(|t - 8| \geq 3\)

(4) differs from \(-6\) by at most 14.

SOLUTION: \(|z - (-6)| \leq 14\) or \(|z + 6| \leq 14\)

(5) \(x\) is less than \(a\) units from \(b\).

SOLUTION: \(|x - b| < a\)

Properties of the Absolute Value:

1. \(|a \cdot b| = |a| \cdot |b|\)

EXAMPLE: (1) \((5)(-3) = |5|\cdot|-3| = 5 \times 3 = 15\)

(2) \(-4x^2 = |-4| \cdot |x^2| = 4x^2\)

Note that \(x^2 \geq 0\) ; thus \(|x^2| = x^2\)

\(x^2 = x^2\) for \(x^2 \geq 0\) and \(x^2 \geq 0\) \(\Box\)

2. \(\left|\frac{a}{b}\right| = \frac{|a|}{|b|}\)

EXAMPLE: (1) \(\left|\frac{8}{-4}\right| = \left|\frac{8}{-4}\right| = \frac{8}{4} = 2\)

(2) \(\left|\frac{-5}{x^2 + 1}\right| = \left|\frac{-5}{x^2 + 1}\right| = \frac{5}{x^2 + 1}\)

(Note: \(x^2 + 1 \geq 0\))

(3) \(\left|\frac{-1}{x}\right| = \left|\frac{-1}{x}\right| = \frac{1}{|x|} \Box\)

3. \(|a - b| = |b - a|\)

EXAMPLE: \(|3 - 7| = |7 - 3| = 4\) \(\Box\)

4. \(-|a| \leq a \leq |a|\)

EXAMPLE: (1) \(-|2| \leq 2 \leq |2|\)

(2) \(-|-8| \leq -8 \leq |-8| \Box\)

EXERCISES 1.4:

In problems 1 – 21, solve the given inequalities:
1. $|8x + 17| \leq 13$
2. $2|3 - 4x| > 3$
3. $|9x - 32| > -14$
4. $|1 - 3x| < \frac{2}{5}$
5. $\left| \frac{2x - 7}{-4} \right| \geq 10$
6. $\left| \frac{x + 3}{15} \right| < \frac{2}{3} - \frac{3}{5}$
7. $(2x - 1)^2 \leq 100$
8. $|37 - 41x| < 0$
9. $|2x + 3| < |1 - 2x|$
10. $|2x + 3| < -12$
11. $-7 < 2 - 11x \leq 1$
12. $4x + 9 \leq 1$
13. $|1 - 2y| > \frac{1}{5}$
14. $\left| \frac{x + 4}{-3} \right| \geq 2$
15. $\left| \frac{1 - p}{3} \right| \leq \frac{5 - 1}{6}$
16. $\frac{-1}{7} \leq 1 - \frac{3x}{2} \leq 1$
17. $|4t - 21| \leq -8$
18. $\frac{2}{\sqrt{x - 3}} \leq \frac{1}{5}$
19. $|1 + 7q| \geq -24$
20. $(3 - 2t)^2 \leq 49$
21. $|x + 1| \leq |3 - x|

In problems 22 – 30, solve the given equations:

22. $|5 - 7y| = 9$
23. $|4z + 3| = -2$
24. $|2x - 3| = |x|
25. $\frac{2}{3} - \frac{y}{2} = 0$
26. $\left| \frac{6}{2x - 1} \right| = 3$
27. $|3x - 8| = 2$
28. $\left| \frac{3}{2 - x} \right| = 4$
29. $|3x + 4| = 5x$
30. $|5x - 6| = x^2$

31. Using the absolute value symbol, express each fact:
   a. $x$ is more than 7 units from $-31$
   b. $y$ is less than 10 units from 22
   c. The distance between $x$ and $3/4$ is less than 2
   d. $a$ Differs from $-2$ by no more than 5 units
   e. $y$ is more than 5 units from 20
   f. $x$ is less than 3 units from $-15$
   g. $t \geq -12$ and $t \leq 12$
   h. $x < -7$ or $x > 7$
   i. The distance between $x$ and $-3$ is no more than 1

32. Classify the following statements as either true or false:
   a. $|x + y| = |x| + |y|$
   b. $|-a - b| = a + b$
   c. $|3 - x| = x^2$
   d. $\left| \frac{a - b}{|a| - |b|} \right| = 1$
   e. $\left| \frac{a + b}{|a| + |b|} \right| = 1$
   f. $\left| \frac{x}{y} \right| = \frac{|x|}{|y|}$
   g. $|a - b| = |a| - |b|$
   h. $|x \cdot y| = |x| \cdot |y|$
   i. $|\pi - 1| = \pi + 1$

33. If $4 < x < 7$, find $|x - 3| - |x - 7|$
34. Evaluate $\left| (3t)^2 \right| + |3t|^2$
35. If $a = -9, b = -7$ and $c = 6$, find $\left| \frac{a - b}{|a| - |b|} \right|$ and $\left| \frac{a - c}{|a| - |c|} \right|$
Chapter 1: Applications of equations and Inequalities

Solved problems of section 1.2(Linear Inequalities):

In problems 12 – 20, solve the given inequalities:

12. \(-7(x + 2) \leq -x\)

Sol. \(-7x - 14 + x \leq 0 \Rightarrow -6x \leq 14 \Rightarrow x \geq \frac{14}{-6} \Rightarrow x \geq \frac{-7}{3}\)

∴ Solution set \([-7/3, \infty)\)

13. \(2(4 - x) - 1 > 3(x + 3) - 5(2 - x)\)

Sol. \(8 - 2x - 1 > 3x + 9 - 10 + 5x \Rightarrow 7 - 2x > 8x - 1 \Rightarrow -10x > -8 \Rightarrow x < -8/10\)

∴ \(x < 4/5 \Rightarrow \text{Solution set } (-\infty, 4/5]\)

14. \(\frac{3x^2 - 7x}{3} < x^2 + \frac{3x - 1}{-2}\)

Sol. \(6x^2 - 14x < 6x^2 - 9x + 3 \Rightarrow -5x < 3 \Rightarrow x > -3/5\)

∴ Solution set \((-3/5, \infty)\)

15. \(\frac{y + 1}{6} \leq \frac{-3}{2} - \frac{5 + y}{3}\)

Sol. \(y + 1 \leq -9 - 2(5 + y) \Rightarrow 3y \leq -20 \Rightarrow y \leq -20/3\)

16. \(\frac{3}{2}x + 2 < \frac{7}{6}x - 1\)

Sol. \(9x + 12 < 7x - 6 \Rightarrow 2x < -18 \Rightarrow x < -9\)

17. \(4s - 1 \geq -2(3 - 2s) + 5\)

Sol. \(4s - 1 \geq -6 + 4s + 5 \Rightarrow 4s - 4s \geq -6 + 5 + 1 \Rightarrow 0 \geq 0 \text{ (the result is always true)}\)

∴ Solution set = \(\emptyset\)

18. \(3x - 7 \geq x - 2(3 - x)\)
Sol. \(3x - 7 \geq x - 6 + 2x \Rightarrow 3x - x - 2x \geq -6 + 7 \Rightarrow 0 \geq 1\) (the result is always false)
\[\therefore\text{ Solution set } = \emptyset\]

19. \(2(-3t - 5) + 11 > 5 - 2(6 + 2t)\)
Sol. \(-6t - 10 + 11 > 5 - 12 - 4t \Rightarrow -6t + 1 > -7 - 4t \Rightarrow -2t > -8\)
\[\therefore t < -8/2 \Rightarrow t < 4\]

20. \((1 - 3x)^2 - 9(2 - x)^2 \geq 5\)
Sol. \(1 - 6x + 9x^2 - 9(4 - 4x + x^2) \geq 5 \Rightarrow 1 - 6x + 9x^2 - 36 + 36x - 9x^2 \geq 5\)
\[30x - 35 \geq 5 \Rightarrow 30x \geq 40 \Rightarrow x \geq 4/3\]

21. Suppose that \(a < 0\) and \(b > 0\). Classify the following expressions as either true or false:

\[\begin{array}{ll}
a. & ab > 0 \quad \text{False: } b > 0 \text{ and } a < 0 \Rightarrow ab < 0 \Rightarrow ab < 0 \\
b. & a \leq b \quad \text{True: } a < 0 < b \Rightarrow a < b \Rightarrow a \leq b \\
c. & b^2 \geq b \quad \text{False: } \text{Let } b = 1/2 \Rightarrow b^2 = 1/4 < 1/2 = b \Rightarrow b^2 < b \\
d. & a/b < 0 \quad \text{True: } \text{Let } a = 1 \Rightarrow a/b < a \Rightarrow a/b < 0 \\
e. & a - b < 0 \quad \text{True: } a < 0 < b \Rightarrow a < b \Rightarrow a - b < 0 \\
f. & a^2 > a \quad \text{True: } a^2 \geq 0 > a \Rightarrow a^2 > a \\
\end{array}\]

Solved problems of section 1.4 (Absolute value):

In problems 12 – 21, solve the given inequalities:

12. \(|4x + 9| \leq 1\)
Sol. \(|4x + 9| \leq 1 \Rightarrow -1 \leq 4x + 9 \leq 1 \Rightarrow -1 - 9 \leq 4x \leq 1 - 9 \Rightarrow -10 \leq 4x \leq -8\)
\[\begin{align*}
-10/4 & \leq x \leq -8/4 \\
\therefore \text{ Solution set } & = [-5/2, -2]
\end{align*}\]

13. \(|1 - 2y| > \frac{1}{5}\)
Sol. Case 1: \(1 - 2y > \frac{1}{5} \Rightarrow 5 - 10y > 1 \Rightarrow -10y > -4 \Rightarrow y < \frac{-4}{-10} \Rightarrow y < \frac{2}{5}\)
Case 2: \(1 - 2y < -\frac{1}{5} \Rightarrow 5 - 10y < -1 \Rightarrow -10y < -6 \Rightarrow y > \frac{-6}{-10} \Rightarrow y > \frac{3}{5}\)
\[\therefore \text{ Solution set } = (-\infty, 2/5) \cup (3/5, \infty)\]

14. \(\left|\frac{x + 4}{-3}\right| \geq 2\)
Sol. Case 1: \(\frac{x + 4}{-3} \geq 2 \Rightarrow x + 4 \leq -6 \Rightarrow x \leq -10\)
Case 2: \( \frac{x+4}{-3} \leq -2 \) \( \Rightarrow \) \( x+4 \geq 6 \) \( \Rightarrow \) \( x \geq 2 \)

\[
\therefore \text{ Solution set } = (-\infty,-10]\cup[2,\infty)
\]

15. \( \left| \frac{1-p}{3} \right| \leq \frac{5}{6} - \frac{1}{2} \)

Sol. \( 2\left| 1-p \right| \leq 5 - 3 \)

\[
2\left| 1-p \right| \leq 2 \Rightarrow \left| 1-p \right| \leq 1 \Rightarrow -1 \leq 1-p \leq 1 \Rightarrow -2 \leq -p \leq 0 \Rightarrow 2 \geq p \geq 0
\]

\[
\therefore \text{ Solution set } = [0,2]
\]

16. \( \frac{-1}{7} \leq \frac{1-3x}{2} \leq 1 \)

Sol. \( -2 \leq 7(1-3x) \leq 14 \)

Multiplying both sides by 14

\[
-2 \leq 7 - 21x \leq 14 \Rightarrow -9 \leq -21x \leq 7 \Rightarrow -9/(21) \geq x \geq 7/(21)
\]

\[
\therefore 3/7 \geq x \geq -1/3 \Rightarrow \text{ Solution set } = \left[ -\frac{1}{3}, \frac{3}{7} \right]
\]

17. \( |4t-21| \leq -8 \)

Sol. \( \therefore \text{ Solution set } = \emptyset \) (The absolute value is always greater than any negative number)

18. \( \frac{2}{|x-3|} \leq \frac{1}{5} \)

Sol. \( 10 \leq |x-3| \)

Multiplying both sides by the L.C.D. = 5\(|x-3|\)

Case 1: \( x-3 \geq 10 \) \( \Rightarrow \) \( x \geq 13 \)

Case 2: \( x-3 \leq -10 \) \( \Rightarrow \) \( x \leq -7 \)

\[
\therefore \text{ Solution set } = (-\infty,-7]\cup[13,\infty)
\]

19. \( |1+7q| \geq -24 \)

Sol. \( \therefore \text{ Solution set } = \square \) (The absolute value is always greater than any negative number)

20. \( (3-2t)^2 \leq 49 \)

Sol. \( \sqrt{(3-2t)^2} \leq \sqrt{49} \Rightarrow 3-2t \leq 7 \Rightarrow -7 \leq 3-2t \leq 7 \Rightarrow \ -10 \leq -2t \leq 4 \)

\[
\therefore 5 \geq t \geq -2 \Rightarrow \text{ Solution set } = [-2,5]
\]

21. \( |x+1| \leq |3-x| \)

Sol. \( (|x+1|)^2 \leq (|3-x|)^2 \Rightarrow (x+1)^2 \leq (3-x)^2 \Rightarrow x^2 + 2x + 1 \leq 9 - 6x + x^2 \)

\[
\therefore 8x \leq 8 \Rightarrow x \leq 1
\]

In problems 27 – 30, solve the given equations:
27. $|3x - 8| = 2$
Sol. Case 1: $3x - 8 = 2 \Rightarrow 3x = 10 \Rightarrow x = 10/3$
Case 2: $3x - 8 = -2 \Rightarrow 3x = 6 \Rightarrow x = 2$
∴ Solution set = $\{10/3, 2\}$

28. $\left| \frac{3}{2-x} \right| = 4$
Sol. Case 1: $\frac{3}{2-x} = 4 \Rightarrow 3 = 4(2-x) \Rightarrow 3 = 8 - 4x \Rightarrow 5 = -4x \Rightarrow x = 5/4$
Case 2: $\frac{3}{2-x} = -4 \Rightarrow 3 = -4(2-x) \Rightarrow 3 = -8 + 4x \Rightarrow 11 = 4x \Rightarrow x = 11/4$
∴ Solution set = $\{5/4, 11/4\}$

29. $|3x + 4| = 5x$
Sol. Case 1: $3x + 4 = 5x \Rightarrow 3x - 5x = -4 \Rightarrow -2x = -4 \Rightarrow x = 2$
Case 2: $3x + 4 = -5x \Rightarrow 3x + 5x = -4 \Rightarrow 8x = -4 \Rightarrow x = -1/2$
Checking case 2 shows that $x = -1/2$ is impossible.
∴ Solution set = $\{2\}$

30. $|5x - 6| = x^2$
Sol. Case 1: $5x - 6 = x^2 \Rightarrow x^2 - 5x + 6 = 0 \Rightarrow (x-2)(x-3) = 0 \Rightarrow x = 2, x = 3$
Case 2: $5x - 6 = -x^2 \Rightarrow x^2 + 5x - 6 = 0 \Rightarrow (x-6)(x+1) = 0 \Rightarrow x = -6, x = 1$
∴ Solution set = $\{1, 2, 3, -6\}$

31. Using the absolute value symbol, express each fact:

<table>
<thead>
<tr>
<th>Problem</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>e. $y$ is more than 5 units from 20</td>
<td>$</td>
</tr>
<tr>
<td>f. $x$ is less than 3 units from -15</td>
<td>$</td>
</tr>
<tr>
<td>g. $t \geq -12$ and $t \leq 12$</td>
<td>$</td>
</tr>
<tr>
<td>h. $x &lt; -7$ or $x &gt; 7$</td>
<td>$</td>
</tr>
<tr>
<td>i. The distance between $x$ and $-3$ is no more than 1</td>
<td>$</td>
</tr>
</tbody>
</table>

32. Classify the following statements as either true or false:

<table>
<thead>
<tr>
<th>Problem</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>e. $\frac{</td>
<td>a</td>
</tr>
<tr>
<td>f. $\left</td>
<td>\frac{x}{y} \right</td>
</tr>
<tr>
<td>g. $</td>
<td>a - b</td>
</tr>
<tr>
<td>h. $</td>
<td>x \cdot y</td>
</tr>
</tbody>
</table>
i. \(|-\pi - 1| = \pi + 1\) True: \(|-\pi - 1| = |(\pi + 1)| = |-1||\pi + 1| = \pi + 1

33. If \(4 < x < 7\), find \(|x-3|-|x-7|\)
Sol. \(x > 4 \Rightarrow x > 3 \Rightarrow x-3 > 0 \Rightarrow |x-3| = x-3\)
\(x < 7 \Rightarrow x-7 < 0 \Rightarrow |x-7| = -(x-7) = 7-x\)
\(\therefore |x-3|-|x-7| = (x-3)-(7-x) = x-3-7+x = 2x-10\)

34. Evaluate \(|(3t)^2|+|-3t|^2\)
Sol. \(|(3t)^2|+|-3t|^2 = (3t)^2 + (-3t)^2 = 9t^2 + 9t^2 = 18t^2\)

35. If \(a = -9, b = -7\) and \(c = 6\), find \(\frac{|a-b|}{|a|-|b|}\) and \(\frac{|a-c|}{|a|-|c|}\)
Sol. \(\frac{|a-b|}{|a|-|b|} = \frac{|-9-(-7)|}{|-9|-|-7|} = \frac{2}{-2} = -1\)
\(\frac{|a-c|}{|a|-|c|} = \frac{|-9-6|}{|-9|-|-6|} = \frac{15}{3} = 5\)