Section 3.5 – IS – LM Analysis

Questions

(a) In a simple economy, Real Money Demand: \( MD = 120 + 0.75Y - 900R \) where

\( Y \) is real aggregate income and \( R \) is the rate of interest expressed as a decimal.

Real Money Supply: \( MS = 1125 \)

Find the equation for the LM curve.

(b) This economy has no government or external trade. Its goods market is modeled by the equations

Consumption: \( C = 150 + 0.7Y \)

Investment: \( I = 330 - 1200R \)

Write down the condition for equilibrium in the goods market and obtain an expression for the IS curve.

(c) Find the equilibrium values of \( Y, R, C \) and \( I \) for this economy.

(d) Suppose that autonomous investment increases by 39. What are the effects of this change on the equations and the equilibrium values?

2. The following equations describe an imaginary economy that has no government:

\[ C = 521 + 0.75Y \] Consumption

\[ I = 874 - 925R \] Investment, where the rate of interest, \( R \), is a decimal

\[ MD = -1140R + 0.8Y \] Money demand

\[ MS = 4136 \] Money supply

(a) Find equations for the IS and LM curves. What are the equilibrium values of \( Y \) and \( R \)? Check that injections equal withdrawals.
(b) What are the effects of a reduction in the supply of money to $MS = 4095$?

3. An imaginary economy is described by the equations:
   
   \[ C = 130 + 0.8Yd \] Consumption
   \[ Yd = Y - T \] Disposable income
   \[ T = 70 \] Lump sum tax
   \[ G = 110 \] Government expenditure
   \[ I = 680 - 1200R \] Investment, where the rate of interest, $R$, is a decimal
   \[ MD = -960R + 0.6Y \] Money demand
   \[ MS = 2364 \] Money supply

(a) Find equations for the IS and LM curves. What are the equilibrium values of $Y$ and $R$?

Is the government budget in deficit or surplus?

(b) What is the effect of an increase in taxation to $T = 108$?

4. An economy with a government but no external trade is modeled by the equations
   
   Consumption: \[ C = 30 + 0.75Yd \]
   Investment: \[ I = 570 - 2500R \] where $R$ is the rate of interest expressed as a decimal.
   Government expenditure: \[ G = 400 \]
   Income tax, levied on all income: \[ t = 0.2 \]
   Real Money Demand: \[ MD = 8 + 0.8Y - 1200R \]
   Real Money Supply: \[ MS = 1760 \]
Answers

1. (a) For money market equilibrium MD = MS. Substituting for these gives

   \[ 120 + 0.75Y - 900R = 1125 \]

   Adding 900R – 120 to both sides we find

   \[ 0.75Y = 1005 + 900R \]

   Dividing by 0.75 gives an expression for Y from the money market equilibrium condition

   \[ Y = 1340 + 1200R \]

   This is the LM equation.

(b) For equilibrium in the goods market Y = AD and we have

   \[ AD = C + I \]

   Substituting for C and I we obtain terms in Y and R on the right-hand side

   \[ AD = 150 + 0.7Y + 330 - 1200R = 480 + 0.7Y - 1200R \]

   Substituting in the equilibrium condition gives

   \[ Y = 480 + 0.7Y - 1200R \]

   Subtracting 0.7Y from both sides we find

   \[ Y = 480 - 1200R \]

   Dividing by 0.3 we obtain an expression for Y in terms of R, namely

   \[ Y = 1600 - 4000R \]

   This is the equation of the IS curve.

(c) Overall macroeconomic equilibrium requires that the values of Y and R satisfy the conditions for equilibrium in both markets. The LM equation and the IS equation...
constitute a pair of simultaneous equations in two unknowns, \( Y \) and \( R \). We have the equations

\[
Y_{LM} = 1340 + 1200R \quad \text{(LM equation)}
\]

\[
Y_{IS} = 1600 - 4000R \quad \text{(IS equation)}
\]

adding subscripts to the Ys to identify the equation to which each belongs. For equilibrium in both markets,

\[
Y_{LM} = Y_{IS}
\]

and substituting the right hand sides gives

\[
1340 + 1200R = 1600 - 4000R
\]

Collecting terms, we have

\[
5200R = 260
\]

\[
R = 0.05
\]

Substituting \( R \) back into the LM equation gives

\[
Y_{LM} = 1340 + (1200 \cdot 0.05) = 1400
\]

and we can confirm this by substituting for \( R \) in IS

\[
Y_{IS} = 1600 - (4000 \cdot 0.05) = 1400
\]

Equilibrium \( Y \) and \( R \) are 1400 and 0.05 respectively.

Substituting in the consumption and investment functions gives

Consumption: \( C = 150 + 0.7(1400) = 1130 \)

Investment: \( I = 330 - 1200(0.05) = 270 \)

(d) An increase of 39 in autonomous investment changes the investment function to
\[ I_2 = 369 - 1200R \]

For goods market equilibrium we now require
\[ Y = 519 + 0.7Y - 1200R \quad \text{or} \quad 0.3Y = 519 - 1200R \]

The new IS curve is therefore
\[ Y_{IS2} = 1730 - 4000R \]

The LM curve is unchanged by the rise in autonomous investment, so overall equilibrium is achieved when
\[ Y_{LM} = Y_{IS2} \]

and substituting the expressions gives
\[ 1340 + 1200R = 1730 - 4000R \]

Collecting terms we have
\[ 5200R = 390 \]

and so we find that at the higher level of investment \( R = 0.075 \). Substituting into the LM equation gives
\[ Y_{LM} = 1340 + (1200 \times 0.075) = 1430 \]

The new equilibrium is at \( Y = 1430 \), \( R = 0.07 \)

Substituting in the consumption and investment functions gives

Consumption:
\[ C = 150 + 0.7(1430) = 1151 \]

Investment:
\[ I = 369 - 1200(0.075) = 279 \]
With the increase in autonomous investment, income and consumption have increased and the interest rate has risen. The increase in the interest rate chokes off induced investment so that it only rises by 9. The overall increase is less than the change in autonomous investment, because of the rise in the interest rate.

2. (a) \( Y = 5580 - 3700R \) is the equation of the IS curve
\[ Y = 5170 + 1425R \] is the equation of the LM curve
\[ R = 0.08, \ Y = 5284, \ C = 4484, \ S = 800, \ I = 800, \ W = J \]
(b) \( YLM2 = 5118.75 + 1425R \) is the new LM equation
\[ R = 0.09, \ Y = 5247, \ C = 4456.25, \ S = 790.75, \ I = 790.75, \ W = J \]

3. (a) \( Y = 4320 - 6000R \) is the equation of the IS curve
\[ Y = 3940 + 1600R \] is the equation of the LM curve
\[ R = 0.05, \ Y = 4020, \ T - G = -40, \text{ government budget deficit} \]
(b) \( Y = 4168 - 6000R \) is the new IS equation
\[ R = 0.03, \ Y = 3988, \ T - G = -2, \text{ reduced government budget deficit} \]

4. (a) \( Y = 2500 - 6250R \) is the equation of the IS curve
\[ Y = 2190 + 1500R \] is the equation of the LM curve
\[ R = 0.04, \ Y = 2250, \ Yd = 1800, \ C = 1380, \ S = 420, \ I = 470, \ T = 450 \]
\[ J = I + G = 870, \ W = S + T = 870 \text{ so } J = W. \]
(b) \( Y = 2655 - 6250R \) is the new IS equation
\[ R = 0.06, \ Y = 2280, \ Yd = 1824, \ C = 1398, \ S = 426, \ I = 420, \ T = 456 \]
\[ J = I + G = 882, \ W = S + T = 882 \text{ so } J = W. \]
Section 3.2 – Notation and Model

Specification

Questions

1. A simple economy with no government has the consumption function .
   \[ C = 64 + 0.7Y \]
   Find an expression for the saving function. What level of income \( Y \) would generate saving of 500?

2. An economy with a government that levies a lump sum tax, \( T \), has \( Y = 1900 \) and
   \[ C = 50 + 0.75 \text{Yd} \]
   \( I = 400 \)
   \( T = 300 \)
   \( G = 250 \)
   (a) Write down expressions for disposable income and the saving function and .find their values
   (b) What is the government’s budget surplus or deficit?
   (c) What is the surplus or deficit in the private sector’s financial balance?

3. An open economy with no government has the consumption function
   \[ C = 120 + 0.9Y \]
   Imports are given by
   \[ Z = 30 + 0.2Y \]
I = 770, \( X = 580 \) and \( Y = 4800 \)

(a) What is the marginal propensity to consume?

(b) What is the foreign trade surplus?

(c) Show whether \( W = J \)
Answers

1. \[ S = Y - C = Y - (64 + 0.7Y) \]
   \[ = -64 + 0.3Y \]
   If \( 500 = -64 + 0.3Y \) \( 0.3Y = 564 \) and
   \( Y = 1880 \) to generate saving of 500.

2. (a) \( Yd = Y - T = 1600 \)
   \( C = 1250 \)
   \( S = Yd - C = 1600 - 1250 = 350 \)
   (b) \( T - G = 50 = \) government budget surplus.
   (c) \( S - I = -50 = \) surplus in the private sector’s financial balance or deficit of 50.

3. (a) Marginal propensity to consume = 0.9
   (b) \( Z = 990 \)
       Foreign trade surplus = \( X - Z = -410 \)
   (c) \( C = 4440 \)
       \( S = 360 \)
       \( W = S + Z = 1350 \)
       \( J = I + X = 1350 = W \)
Section 3.4 – Keynesian Cross Model

Questions

1. An economy is specified by the equations
   
   \[ C = 375 + 0.8Yd \text{ Consumption expenditure} \]
   
   \[ T = 400 \text{ Lump sum direct taxation} \]
   
   \[ I = 520 \text{ Investment expenditure} \]
   
   \[ G = 450 \text{ Government expenditure on goods and services} \]
   
   Find equilibrium income. Are injections equal to withdrawals?

2. The consumption function for a closed economy is given by .
   
   \[ C = 410 + 0.75Yd \]
   
   There is a proportional income tax at a rate of \( t = 0.2 \)
   
   The government spends 570 on goods and services and private investment is 480.
   
   (a) Find equilibrium income and the government budget surplus.
   
   (b) Do these change if the government increases its expenditure by 230?

3. An open economy with no government has the consumption function.
   
   \[ C = 180 + 0.7Y \]
   
   Imports are given by
   
   \[ Z = 20 + 0.1Y \]
   
   \[ I = 650 \text{ and } X = 630 \]
   
   (a) Find the equilibrium income.
   
   (b) What is the foreign trade surplus?
   
   (c) Show whether \( W=J \).

Mathematics for Economics and Business: An Interactive Introduction Section 3.4
**Answers**

1. \( Y = AD = C + I + G \) in equilibrium and so
   \[ Y = 375 + 0.8Yd + 520 + 450 = 1345 + 0.8Yd \]
   \[ Yd = Y - T \] so
   \[ Y = 1345 + 0.8(Y - 400) = 1345 + 0.8Y - 320 \]
   \[ 0.2Y = 1025 \]
   \[ Y = 5125 \] is the equilibrium income
   \[ Yd = 4725 \]
   \[ C = 4155 \]
   \[ S = 570 \]
   \[ W = S + T = 970 \]
   \[ J = I + G = 970 = W \]

2. (a) \[ C = 410 + 0.75Yd \]
   \[ Yd = Y - 0.2Y = 0.8Y \]
   \[ Y = C + I + G \] in equilibrium and so
   \[ Y = 410 + 0.6Y + 480 + 570 \]
   \[ 0.4Y = 1460 \]
   \[ Y = 3650 \] is the equilibrium income
   \[ T = 0.2(3650) = 730 \]
   \[ T - G = 160 \] is the government budget surplus

(b) If \( G = 800, 0.4Y = 1690 \)
\( Y = 4225 \) is the equilibrium income

\( T = 0.2(4225) = 845 \)

\( T - G = 45 \) is the government budget surplus

3. (a) \( Y = C + I + X - Z \) in equilibrium and so
   \[
   Y = 180 + 0.7Y + 650 + 630 - (20 + 0.1Y)
   \]
   \[
   0.4Y = 1440
   \]
   \( Y = 3600 \) is the equilibrium income

(b) \( Z = 20 + 0.1Y = 380 \)
   \( X - Z = 630 - 380 = 250 \) is the foreign trade surplus

(c) \( C = 180 + 0.7Y = 2700 \)
   \( S = Y - C = 900 \)
   \( W = S + Z = 900 + 380 = 1280 \)
   \( J = I + X = 650 + 630 = 1280 = W \)
Section 4.3 – Simple and Compound Interest

Questions

1. What value of R would you substitute in the compound interest formula for each of the following annual interest rates?
   a) 12%
   b) 8.5%
   c) 6.24%

2. Over how many time periods do we calculate the interest payments if compounding is:
   a) monthly for 5 years?
   b) Quarterly for 7 years?
   c) Weekly for 3 years?
   d) Monthly for 2.75 years?

3. If the time period to which an interest rate applies is not stated explicitly, it is assumed to be:
   a) 1 year
   b) 2 years
   c) 1 quarter
   d) 1 week

4. If an annual interest rate of 12% is compounded quarterly the value of R in the compound interest formula is:
   a) 0.12
b) 0.48

c) 0.03

d) 0.04

5. Arrange the amounts of interest paid in order of ascending size if lenders offer to lend you money for 3 years at these different interest rates:

a) at 16% compounded continuously

b) at 16% simple interest

c) at 4% per quarter compounded quarterly

d) at 16% compound interest

6. (a) A man borrows $5,000 for 4 years at an annual rate of compound interest of 6.2%. How much does he owe at the end of the 4 year period?

(b) How much would he owe if the interest was simple as opposed to Compound?

7. You inherit $3,300 which you plan to save for 3 years whilst you are at university.

a) Would you prefer to put it in an account offering 8% simple interest or one offering 7% compound interest?

b) How much would you inheritance be worth if you found an account offering a rate of 7% compounded continuously?

8. Find the future value after 4 years of a principal of $2,000 which earns interest at a rate of 4% every 6 months, compounded twice a year.

9. If a loan of $1,000 for a car is offered by a loan shark at a rate of interest of 35% over a period of 1 year. Calculate the final outstanding debt if the interest is
10. How much are your savings of $1,200 worth in 2 years time if the account offers you an annual interest rate of 4%
   a) Compounded monthly?
   b) Compounded quarterly?
   c) Compounded annually?

11. You invest capital of $10,000 in an account offering 6% annual interest. How much would your money be worth after 5 years if the interest was
   a) simple?
   b) Compounded annually?
   c) Compounded continuously?

12. Your unagreed overdraft is piling on interest at such a rate that you decide to get a loan to pay it off. You currently owe $900 and your account adds compound interest of 2% every month. You are considering a loan which offers a quarterly rate of 6% compounded continuously. Would this be a wise choice?

13. An accountant finds a discrepancy in a firm's accounts. They predicted that their savings of $4,000 five years ago would now be worth $5352.90 in an account bearing interest of 6%, but in fact there is only $5200. How could they have miscalculated by such a large amount?

14. (a) When you take out a loan, is it better to borrow money where the interest...
rate is calculated continuously or annually?

(b) When you open a savings account, is it better to choose an account where
the interest is calculated continuously or annually?

15. Write down the formula for calculating
   a) simple interest
   b) compound interest
   c) continuous compound interest

**Answers**

1. (a) 0.12
   (b) 0.085
   (c) 0.0624

2. (a) 60
   (b) 28
   (c) 156
   (d) 33

3. The correct answer is (a)

4. The correct answer is (c)

5. The correct order is (b), (d), (c), (a)

6. (a) $6360.16
   (b) $6240.00

7. (a) At a simple rate of 8%: $4092.00; at a compound rate of 7%: $4042.64 therefore
   the simple rate is better.
   (b) $4071.14

8. $2737.14
9. (a) $1398.68
(b) $1411.98
(c) $1419.07

10. (a) $1299.77
(b) $1299.43
(c) $1297.92

11. (a) $13,000
(b) $13,382.26
(c) $13,498.59

12. After 1 year the overdraft would have increased to $1141.42. After 1 year, the amount owing on the loan would be $1144.12. The loan is marginally worse than the overdraft, so would not be a wise choice.

13. The prediction of $5352.90 was calculated under the assumption that the interest would be compounded whereas the account in fact only gave simple interest.

14. (a) Annually
(b) Continuously

\[
(1 + nR) V_0 = V_a \hspace{1cm} (a)
\]
\[
(1 + R)^n V_0 = V_c \hspace{1cm} (b)
\]
\[
V_0 e^{Rt} = V_b \hspace{1cm} (c)
\]
Section 4.8 – Net Present Value

Questions

1. A firm is considering buying a new computer system. The initial cost would be $530,000. The new system would generate returns in the three years following its implementation of $130,000 in year 1, $360,000 in year 2 and $240,000 in year 3. The discount rate is 4%. Using units of $'000, answer the questions below:
   a) What are the discount factors for years 0, 1, 2 and 3?
   b) What is the present value of returns for year 2?
   c) What is the NPV of the project?
   d) Should the project be undertaken?
   e) If the discount rate was in fact 11%, should the project be undertaken in these circumstances?

2. A company is considering a new warehousing complex, the initial cost being $910,000. The complex would generate returns in the four years following its coming into use of $210,000 in year 1, $350,000 in year 2, $420,000 in year 3 and $140,000 in year 4.
   a) Using a 5% discount rate, show whether the firm should invest in the new complex.
   b) If the appropriate discount rate is in fact 8%, show whether this alters your recommendation. Does it change if the discount rate is 10%?
   c) Estimate the approximate internal rate of return of this project.

3. A factory wishes to invest in some new machinery. The initial outlay is
\$450,000 and would generate returns in year 1 of \$50,000, \$250,000 in year 2 and \$230,000 in year 3. The appropriate discount rate is 8%.

a) What is the discount factor in year 3?

b) What are the present values of the returns in years 1 and 2?

c) What is the NPV of the project?

d) Should the project be undertaken? If the discount rate were 6%, would you recommend the investment in new machinery?

e) What is the IRR of the project?

4. The start-up cost for a new business is estimated at \$50,000. It would generate returns in year 1 of \$5,000, \$20,000 in year 2, \$20,000 in year 3 and \$15,000 in year 4.

a) Using a discount rate of 6%, show whether the entrepreneurs should start up their business.

b) If the appropriate discount rate is in fact 7%, show whether this alters your recommendation.

c) Estimate the IRR of this project.

**Answers**

1. (a) 1, 0.96, 0.92, 0.89
   
   (b) 332.84
   
   (c) 141.20
   
   (d) Yes, because 141.20 > 0
   
   (e) Yes, because NPV = 54.79 > 0

2. (a) NPV = 85.45 which is >0, so the project should go ahead.
   
   (b) NPV = 20.83, so the project should still go ahead; NPV = -18.66, so the project should not be undertaken.
(c) 9%, exact value is 9.038%

3. (a) 0.7938
   (b) 46.30, 214.33
   (c) -6.79
   (d) No, because NPV is negative; Yes, because NPV = 12.78, >0.
   (e) 7.291%, or approximately 7%

4. (a) NPV = 1.19, so the project can go ahead.
   (b) With a discount rate of 7%, the project should not go ahead as NPV = -0.09
   (c) IRR = 6.929%, just below 7% which gives NPV very close to zero

Section 4.10 – Savings and Loans with

Regular Payments

Questions

1. You put $600 given to you on your birthday into a savings scheme offering an interest rate of 1.2% compound per quarter, and you make additional payments into the savings scheme of $150 each quarter.

What is the value of your savings one year later?

Show that when your 15th quarterly payment is made, the amount in the savings scheme exceeds $3000.

Sinking fund

2. You save $120 per month for 2 years at an interest rate of 0.3% compound per month. What is the value of your savings at the end of the 2 years?
Annuity

3. A professor taking early retirement has a lump sum of $119,865 available to purchase an annuity. If the interest rate is 5.1% compound per year, and she wishes to receive an annual annuity payment for 20 years, what is the annual annuity payment that she can purchase?

4. What is the value of a perpetuity that pays $200 per quarter with no time limit if the rate of interest is 1% compound per quarter?

5. What is the value of an annuity that pays $2000 every year for 40 years? Assume interest is at a compound rate of 6% annually.

Mortgage Repayments

6. What is the annual repayment on a $82,500 mortgage over a 20 year period if the interest rate is 5.7%?

   If the amount you repay is divided into equal monthly amounts, what is your monthly repayment?

   What is the capital recovery factor used in this calculation?

7. What is the annual repayment on a $49,000 mortgage over 25 years at a compound annual interest rate of 6%?

   If instead the interest rate is 8%, what is the annual repayment?

Answers

1. After 1 year, \( n = 4 \)

\[
= 1240.21 \left( 150 \left( \frac{1 + 0.012}{1 + 0.012} \right) \right) / 0.012 + 0.012 \right) \left( \frac{1 + 0.012}{1 + 0.012} \right) \frac{1}{4} 600 \left( \frac{1}{4} \right)
\]

This is the amount when interest has been compounded quarterly for a year and your 4th payment of $150 has just been received.
\[ V_{15} = \frac{600 \left(1 + 0.012\right)^{15}}{0.012} \]

When you add your 15th quarterly payment you have $3166.75 in the saving scheme. Immediately before it is added, therefore, you have savings of

$3166.75 - $150 = $3016.75

2. \( N = 24 \) \( R = 0.003 \)

\[ \frac{12 \times (1.003)^3 \times [(1.003)^{24} - 1]}{0.003} = \frac{V}{24} \times \frac{W(1+R)^24-R^2-1}{R} \]

\[ = $2990.53 \]

3. Rearranging the formula \( A = \)

\[ \frac{119,865 \times 0.051}{1 - \left(1.051\right)^{-20}} = \]

\[ = $20,000 \]

\[ A \] \[ = V_{0} \]

\[ (A[1 - (1 + R)^{-n}])/R = (2000[1 - (1.06)^{-40}])/0.06 = $30,092.59 \]
6. \( W = \frac{82,500 \times 0.057}{1 - 1.057^{-20}} = \$7018.55 \quad MR = \begin{pmatrix} 1 - (1 + R)^{-n} \end{pmatrix} \)

Dividing by 12 gives monthly repayments of $584.88

\[ R = \frac{1}{1 - (1 + R)^{-12}} \]

Capital recovery factor =

(Multiply this by the amount you borrow to get your annual repayment)

7. \( W = \frac{49,000 \times 0.06}{1 - 1.06^{-25}} = \$3833.11 \quad MR = \begin{pmatrix} 1 - (1 + R)^{-20} \end{pmatrix} \)

\[ W = \frac{49,000 \times 0.06}{1 - (1 + R)^{-20}} = \$4590.26 \]

Section 5.3 – The Basic Rules of Differentiation

1. Find \( dy/dx \) for each of the following functions

(a) \( y = x^5 \)  
(b) \( y = 2x^4 \)  
(c) \( y = -4x^6 \)  
(d) \( y = 130 \)  
(e) \( y = 4x \)  
(f) \( y = x^7 \)  
(g) \( y = 9x^2 \)  
(h) \( y = -4x^5 \)  
(i) \( y = 2 \)  
(j) \( y = 132x \)  
(k) \( y = x^2 \)  
(l) \( y = 5x^4 \)  
(m) \( y = -3x^3 \)  
(n) \( y = x \)
(o) \( y = 43x \)  
(q) \( y = 10x15 \)  
(s) \( y = 6.5 \)  
(r) \( y = -12x4 \)  
(t) \( y = 32x \)

2. Differentiate each function with respect to \( x \)

(a) \( y = 3x-2 \)  
(b) \( y = -5x-1 \)  
(c) \( y = 9x2/3 \)  
(d) \( y = 5x \)  
(e) \( y = -2x3 \)  
(f) \( y = -6x \)  
(g) \( y = x6 \)  
(h) \( y = 5x-3 \)  
(i) \( y = -3x-7 \)  
(j) \( y = 16x1/4 \)  
(k) \( y = 14x2 \)  
(l) \( y = -5x4 \)  
(m) \( y = -8x \)  
(n) \( y = x7 \)  
(o) \( y = x10 \)  
(p) \( y = -12x3 \)  
(q) \( y = 9x1/3 \)  
(r) \( y = 3x3 \)  
(s) \( y = -4x4 \)  
(t) \( y = 9x13 \)
Answers

\( x^5 \)  
(b) \( 8x^4 \)  
1. (a) 5

(d) 0 \( x^4 \)  
(c) -24

\( x^4 \)  
(e) 4  
(f) 7

\( x^6 \)  
(g) 18x  
(h) -20

(i) 0  
(j) 132

\( x^2 \)  
(k) 2x  
(l) 20

(n) 1 \( x^5 \)  
(m) -9

\( x^{10} \)  
(o) 43  
(p) 11

\( x^8 \)  
(r) -48 \( x^{14} \)  
(q) 150

(s) 0  
(t) 32

\( x^{-3} \)  
(b) 5 \( x^{-2} \)  
2. (a) -6

(d) -5 \( x^{\frac{1}{2}} \)  
(c) 6

\( x^\frac{1}{2} \)  
(f) -3 \( x^{-4} \)  
(e) 6

\( x^{-4} \)  
(h) -15 \( x^3 \)  
(g) 6

\( x^\frac{9}{2} \)  
(j) 4 \( x^{-8} \)  
(l) 21

\( x^\frac{2}{3} \)  
(l) 20 \( x^{-2} \)  
(k) -28

\( x^\frac{1}{2} \)  
(n) 7 \( x^\frac{1}{2} \)  
(m) -4

\( x^{-4} \)  
(p) 36 \( x^9 \)  
(o) 10

\( x^{-4} \)  
(r) -9 \( x^{\frac{2}{3}} \)  
(q) 3

\( x^{\frac{2}{3}} \)  
(t) 3 \( x^{-3} \)  
(s) 16

Section 5.3 – The Basic Rules of Differentiation
1

Questions

1. Find $dy/dx$ for each of the following functions

(a) $y = x^5$  
(b) $y = 2x^4$
(c) $y = -4x^6$  
(d) $y = 130$
(e) $y = 4x$  
(f) $y = x^7$
(g) $y = 9x^2$  
(h) $y = -4x^5$
(i) $y = 2$  
(j) $y = 132x$
(k) $y = x^2$  
(l) $y = 5x^4$
(m) $y = -3x^3$  
(n) $y = x$
(o) $y = 43x$  
(p) $y = x^11$
(q) $y = 10x^15$  
(r) $y = -12x^4$
(s) $y = 6.5$  
(t) $y = 32x$

2. Differentiate each function with respect to $x$

(a) $y = 3x^-2$  
(b) $y = -5x^-1$
(c) $y = 9x^{2/3}$  
(d) $y = 5x$
(e) $y = -2x^3$  
(f) $y = -6x$
(g) $y = x^6$  
(h) $y = 5x^-3$
(i) $y = -3x^-7$  
(j) $y = 16x^{1/4}$
(k) $y = 14x^2$  
(l) $y = 5x^4$
(m) $y = -8x$  
(n) $y = x^7$
(o) $y = x^{10}$  
(p) $y = -12x^3$
(q) $y = 9x^1/3$  
(r) $y = 3x^3$
(s) $y = 4x^4$  
(t) $y = 9x^{13}$
Answers

\(X^5\)  
(b) 8\(X^4\)  
1. (a) 5

d) 0 \(X^5\)  
(c) -24

\(X^6\)  
(e) 4  
(f) 7

\(X^7\)  
(g) 18\(X\)  
(h) -20

(i) 0  
(j) 132

\(X^8\)  
(k) 2\(X\)  
(l) 20

(n) 1 \(X^6\)  
m) -9

\(X^{10}\)  
(o) 43  
(p) 11

\(X^9\)  
(r) -48 \(X^{14}\)  
(q) 150

(s) 0  
(t) 32

\(-X^2\)  
(b) 5\(X^{-3}\)  
2. (a) -6

d) -5 \(X^{\frac{1}{2}}\)  
(c) 6

\(-X^3\)  
(f) -3\(X^{-4}\)  
(e) 6

\(-X^4\)  
(h) -15\(X^6\)  
(g) 6

\(-X^5\)  
(j) 4\(X^{-5}\)  
(l) 21

\(-X^7\)  
(l) 20\(X^{-2}\)  
(k) -28

\(X^8\)  
(n) 7\(X^{\frac{1}{2}}\)  
(m) -4

\(-X^9\)  
(p) 36\(X^9\)  
(o) 10

\(-X^{10}\)  
(r) -9\(X^{\frac{2}{3}}\)  
(q) 3

\(-X^8\)  
(t) 3\(X^{-3}\)  
(s) 16
Section 5.5 – Utility Functions

Questions

1. Sue’s utility function for good X is given by:

\[ U = 60 \]

Find her marginal utility function, and plot it for the values listed to show whether it is diminishing:

\[ X \quad 1 \quad 4 \quad 9 \quad 1 \quad 6 \quad 2 \quad 5 \]

Why are these convenient values to choose?

find the marginal \( x^2 \). Given the total utility function \( U = 15x - 0.5 \) utility function, MU.

3. Which of the following utility functions exhibit diminishing marginal utility, and for which values of \( x \)?

(a) \( U = 7x \)

(b) \( U = 40x - \)

(c) \( U = 12 \)
Mathematics for Economics and Business: An Interactive Introduction Section 5.5

Answers

1. \( MU = 30 \)

\[
\begin{array}{ccccccc}
X & 1 & 4 & 9 & 25 & 36 \\
MU & 30 & 15 & 10 & 6 & 5 \\
\end{array}
\]

This function has diminishing marginal utility for all positive values of \( X \).

Each of the \( X \) values listed is a perfect square, so its square root, \( X^{0.5} \), is an integer. When \( X = 0 \), \( MU \) is infinite.

2. \( MU = \frac{dU}{dx} = 15 - x \)

\( MU \) is a downward sloping line, so it diminishes for all values of \( x \).

3. (a) \( MU = 7 \)

Marginal utility is constant for all values of \( x \).

(b) \( MU = 40 - 2x \)

\( MU \) is a downward sloping line, so it diminishes for all values of \( x \).
(c) \( MU = 9.6 \)

\( MU \) diminishes for all positive values of \( x \), because as \( x \) increases decreases. \( x^{MU} = 1/x^{MU} \)

**Section 5.6 – Revenue Functions**

**Questions**

1. Find the AR and MR functions if total revenue is given by

\[ TR = 120Q - 5 \]

Complete the table of values shown and use them to plot the functions. What is the relationship between average revenue and marginal revenue?

<table>
<thead>
<tr>
<th>( Q )</th>
<th>0</th>
<th>1</th>
<th>10</th>
<th>12</th>
<th>14</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>MR</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>AR</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. If total revenue is given by \( TR = 48Q \) find expressions for AR and MR. In what kind of market might this total revenue function occur?

3. Demand is given by \( P = 33/Q \).

Find expressions for the total revenue and marginal revenue functions.

4. For the demand curve given by \( P = 72 - 0.5Q \), find the total revenue
and marginal revenue functions.

**Answers**

\[
\frac{TR}{Q} = 120 - 5Q \quad \text{1. } AR = TR/Q = (120Q - 5)
\]

\[
MR = \frac{d (TR)}{dQ} = 120 - 10Q
\]

<table>
<thead>
<tr>
<th>Q</th>
<th>0</th>
<th>1</th>
<th>10</th>
<th>12</th>
<th>14</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>MR</td>
<td>120</td>
<td>110</td>
<td>20</td>
<td>0</td>
<td>-20</td>
<td>-</td>
</tr>
<tr>
<td>AR</td>
<td>120</td>
<td>115</td>
<td>70</td>
<td>60</td>
<td>50</td>
<td>0</td>
</tr>
</tbody>
</table>

Average and marginal revenue are both downward sloping linear functions.

The graph shows that MR slopes down twice as steeply as AR.

2. \( TR = 48Q \)

\[
AR = \frac{TR}{Q} = 48, \text{ so AR (or price) is a constant.}
\]

\[
MR = \frac{dTR}{dQ} = 48, \text{ so MR also constant, and the same as AR.}
\]

The demand curve and marginal revenue curve are the same horizontal line. Total revenue is an upward sloping line through the
origin. Curves with these shapes are characteristic of an individual firm operating in a perfectly competitive market.

3. \( TR = P \cdot Q = (33/Q) \cdot Q = 33 \).

This is a constant total revenue function. Whatever output is sold, the total revenue remains at 33.

\[ MR = \frac{d(TR)}{dQ} = 0 \]

As \( Q \), the output traded, increases there is no change in TR and so there is no marginal revenue.

\[ 4. \quad TR = P \cdot Q = (72 - 0.5Q)Q = 72Q - 0.5 \]

\[ MR = \frac{d(TR)}{dQ} = 72 - Q \]

Section 5.7 – Short-Run Production Functions

Questions

1. For the production function \( Q = 10 \), find the marginal product of labour.

2. A firm has the production function

\[ Q = 144L + 12 \]

Find the marginal product of labour function and find its value for the values of \( L \) shown in the table. Sketch the MPL function. Over what range of number of employees does the production function exhibit diminishing marginal returns?
3. Find the marginal product of labour if the production function is

\[ Q = 25 \]

Are there diminishing marginal returns to labour?

Answers

1. MPL = \( dQ/dL = 3 \)

2. MPL = \( dQ/dL = 144 + 24L - 3 \)

There are diminishing marginal returns to labour when \( L > 4 \).

3. MPL = \( dQ/dL = 15 \)
As $L$ increases, so does $\frac{15}{L^Q}$ and as we divide by this larger number $\frac{15}{L^Q}$ decreases. There are diminishing marginal returns to labour.
Section 5.8 – Cost Functions

Questions

1. A firm has the total cost function

\[ TC = 75 + 10Q + 0.25 \]

Find expressions for the firm’s marginal cost, average total cost and average variable cost functions.

Sketch these curves.

2. A firm has the total cost function

\[ TC = 90Q + 170 \]

Find expressions for the firm’s fixed cost, average variable cost and marginal cost functions.

3. A firm has the total cost function

\[ 150 + 160Q – 12 \]

Find the firm’s average total cost, average variable cost and marginal cost functions. Sketch these functions on the same graph.

4. A natural monopoly has the total cost function

\[ Q < 70 \]

\[ TC = 1400 + 420Q – 3 \]

Obtain expressions for its average total cost, average variable cost and marginal cost functions and sketch these curves.
Answers

1. MC = dTC/dQ = 10 + 0.5Q  
   AC = (75 + 10Q + 0.25Q²)/Q = 75/Q + 10 + 0.25Q  
   AVC = (10Q + 0.25Q²)/Q = 10 + 0.25Q
   AVC is an upward sloping line that is always below the AC curve.
   AC falls steeply at first, then slowly, and then rises gently.
   MC is an upward sloping line that cuts AC at its minimum point.

2. FC = constant term = 170  
   AVC = 90Q/Q = 90  
   MC = dTC/dQ = 90  
   AVC and MC coincide: they are the same horizontal line. FC is also a horizontal line, but a different one.

\[
\left(\frac{Q^2}{Q}\right) = 150/Q + 160 - 12Q + 0.5Q^2 + 0.5Q^3  
\]
   AC = (150 + 160Q – 12
AVC = \frac{Q^2}{Q} = 160 - 12Q + 0.5Q^2 + 0.5Q^2

MC = \frac{dTC}{dQ} = 160 - 24Q + 1.5

AC = \frac{1400 + 420Q - 3}{Q} = 1400/Q + 420 - 3Q

AVC = \frac{420Q - 3}{Q} = 420 - 3Q

MC = \frac{dTC}{dQ} = 420 - 6Q

AVC is a downward sloping line. MC is another downward sloping line that slopes down twice as steeply. AC falls steeply at first then falls more slowly getting closer and closer to AVC.
Section 5.9 – Point Elasticity of Demand and of Supply

Questions

1. For the demand function
   \[ P = 56 - 4Q \]
   to 0 at different prices and that when \(-\infty\) show that elasticity varies from
   \[ P=28 \] elasticity is \(-1\).

2. A firm has the demand function
   \[ P = 55 - 2Q \]
   and the supply function
   \[ P = 10 + Q/4 \]
   Find the point price elasticities of demand and supply at the equilibrium price.

3. For the supply function
   \[ Q = 7P - 4 \]
   find the point price elasticity of supply when \(P = 1, 4, 10\)

4. For the demand function
   \[ Q = 64/P \]
   find the quantity demanded, the total revenue and the elasticity at each of the
following prices:

\[ P = 2, 8, 16, 32 \]

5. For the demand function

\[ Q = 50 - 5P \]

find the point price elasticity of demand when \( P = 5 \) and when \( P = 6 \).

If demand now increases so that 10 more units are demanded at any price, what are the new values of elasticity at these prices?

Answers

1. Substituting values of \( P \) we find, for example:

| \( P \) | 56 | 36 | 28 | 24 | 0 |
| \( Q \) | 0  | 5  | 7  | 8  | 14 |

\[ \frac{dP}{dQ} = -4 \text{ so } \frac{dQ}{dP} = -\frac{1}{4} \]

\[ \frac{36}{5} \quad 4 \quad 3 \quad 0 \quad \infty \quad \frac{P}{Q} \]

\[ -\frac{9}{5} \quad -1 \quad -\frac{3}{4} \quad 0 \quad \infty \]

elasticity

2. At equilibrium the same price is read from demand as from supply.

Substituting:

\[ 55 - 2Q = 10 + \frac{Q}{4} \]

\[ 180 = 9Q \]

\( Q = 20 \)

\( P = 15 \)

using demand, \( \frac{dP}{dQ} = -2 \)
demand elasticity = \frac{dQ}{dP} \cdot \frac{P}{Q} = -\frac{1}{2} \cdot \frac{15}{20} = -\frac{3}{8}

using supply, \frac{dP}{dQ} = 1/4

supply elasticity = \frac{dQ}{dP} \cdot \frac{P}{Q} = 4 \cdot \frac{15}{20} = 3

3. if \( P = 1, Q = 3 \)
   
   \( P = 4, Q = 24 \)
   
   \( P = 10, Q = 66 \)
   
   \( \frac{dQ}{dP} = 7 \)
   
   elasticity = \frac{dQ}{dP} \cdot \frac{P}{Q} = 7 \cdot \frac{P}{Q}

if \( P = 1 \) elasticity = 7 \cdot \frac{1}{3} = 2 \frac{1}{3}

\( P = 4 \) elasticity = 7 \cdot \frac{4}{24} = 1 \frac{1}{6}

\( P = 10 \) elasticity = 7 \cdot \frac{10}{66} = 1 \frac{2}{33}

Elasticity gets closer to 1 as \( P \) increases, never goes below 1, so supply is always elastic.

4. \( Q = 32, 8, 4, 2 \)

\( TR = PQ = 64, 64, 64, 64 \)

\( P/Q = 1/16, 1, 4, 16 \)

\( P^2, \frac{dQ}{dP} = -64/P^2 \) Rewrite the demand function: \( Q = 64 \)

elasticity:

\[
\begin{align*}
X \cdot 64^2 & \cdot 4 \cdot X4 \cdot -64/16^2 & \cdot -64/4 \cdot X1/16 & \cdot -64/64 \cdot X1 & \cdot -64/\;
\end{align*}
\]

\[
\begin{align*}
= -1 & \quad = -1 & \quad = -1 & \quad = -1
\end{align*}
\]

5. \(-1, -1.5; \) new demand function is \( Q = 60 -5P, -5/7, -1 \)
Section 5.10 – Investment Multiplier

Questions

1. In a simple economy investment, \( I \), is autonomously determined and the consumption function is

\[
C = 430 + 0.67Y
\]

(a) Find the marginal propensity to consume.

(b) Write down the equilibrium condition if there is no government and no external trade. Find the investment multiplier. Use it to predict the change in \( Y \) if \( I \) increases by 30.

2. An economy with a government that levies a lump sum tax, but no foreign trade is represented by the macroeconomic model

\[
C = 90 + 0.8 \, Y_d
\]

\( I = 800 \)

\( T = 600 \)

\( G = 500 \)

(a) Write down the equilibrium condition and the saving function.

(b) Find the investment multiplier.

(c) Calculate the equilibrium level of national income and show that withdrawals equal injections.

(d) The government replaces the poll tax by a proportional income tax with the same yield. What is the appropriate income tax rate? Calculate the
investment multiplier, and compare its value with its value in (b).

3. An economy with no international trade has the consumption function

\[ C = 30 + 0.8Yd \]

The government levies income tax on all income at a rate of \( t = 0.25 \).

Investment and government spending are autonomous. Find the investment multiplier.

**Answers**

1. (a) \( MPC = \frac{dC}{dY} = 0.67 \)
   
   (b) \( Y = C + I \)
   
   \[
   Y = 430 + 0.67Y + I \quad \text{so} \\
   0.33Y = 430 + I \\
   Y = (430 + I)/0.33
   \]

   \[
   \frac{dY}{dI} = 1/0.33 = 3 \quad \text{This is the investment multiplier.}
   \]

   \[ I = 3(30) = 90 \]

   \[ Y = \frac{dY}{dI} \]

2. (a) \( Y = AD = C + I + G \)
   
   \[
   Y = 90 + 0.8Y - C = 90 + 0.8Y - (90 + 0.8I) = 90 + 0.8Y - 90 - 0.8I - 90Y \\
   S = -90Y \\
   \]

   \[ Y = C + I + G = 90 + 0.8 + I + G \\
   0.2Y = 90 - 0.8T + I + G \quad \text{so} \\
   Y = (90 - 0.8T + I + G)/0.2 \\

   \[
   \frac{dY}{dI} = 1/0.2 = 5 \quad \text{This is the investment multiplier.}
   \]

   (c) \( Y = (90 - 0.8T + I + G)/0.2 = 5(90 - 0.8 \cdot 600 + 800 + 500) = 4550 \)

   \[ = 3950 \]

3. An economy with no international trade has the consumption function

\[ C = 30 + 0.8Yd \]

The government levies income tax on all income at a rate of \( t = 0.25 \).

Investment and government spending are autonomous. Find the investment multiplier.
\[ S = 0.2 \]
\[ W = S + T = 1300 \]
\[ J = I + G = 1300 = W \]

(d) \( tY = T \) so equilibrium income will remain unchanged.

\[ t = T/Y = 600/4550 = 0.1319 \]
\[ Y = C + I + G = 90 + 0.8Y_d + I + G \]

\[ Y = (1 - t) = 0.8681Y \text{ so } Y \]
\[ Y = 90 + 0.8(0.8681Y) + I + G = 90 + 0.6945Y + I + G \]

\[ 0.3055Y = 90 + I + G \text{ so } \]
\[ Y = (90 + I + G)/0.3055 \]

\[ dY/dI = 1/0.3055 = 3.27 \text{ This is the investment multiplier. It is lower} \]
\[ \text{than with a lump sum tax.} \]

3. \( Y = C + I + G = 30 + 0.8Y_d + I + G \)

\[ Y_d = Y (1 - t) = 0.75Y \text{ so } Y \]
\[ Y = 30 + 0.8(0.75Y) + I + G = 30 + 0.6Y + I + G \]

\[ 0.4Y = 30 + I + G \text{ so } \]
\[ Y = (30 + I + G)/0.4 \]

\[ dY/dI = 1/0.4 = 2.5 \text{ This is the investment multiplier.} \]
Section 6.2 – Identifying Maximum and Minimum Turning Points

Questions

1. Identify any maximum or minimum turning points for the functions given

(a) \( y = 2x - \)

(b) \( y = 3x - 2 \)

(c) \( y = \frac{2}{3}x^3 + 3x + 9 \)

(d) \( y = 2x + \frac{1}{x} \)

(e) \( y = \)

(f) \( y = 9 - 2x^2 \)

(g) \( y = 5x - \frac{20}{x} \)

(h) \( y = 10 + 2x^2 \)

(i) \( y = \frac{1}{x} - 2x \)
\( /3 - 10x^3 - 4x^2 \) (j) 4

Answers

1. (a) \( \frac{dy}{dx} = 2 - 2x/3 \) therefore \( x = 3 \) at the turning point.

\[ = -\frac{2}{3} \] which is negative therefore the turning point is a maximum. \( \frac{d^2y}{dx^2} \)

(b) \( \frac{dy}{dx} = 3 \), a constant therefore \( \frac{dy}{dx} \) can never equal zero. A linear function has no turning points.

+ 6x = 2x(x + 3) therefore turning points at \( x = 0 \) and \( x = -3 \). \( \frac{d^2y}{dx^2} \)

(c) \( \frac{dy}{dx} = 2 \)

= 4x + 6 which is positive at \( x = 0 \) (minimum point) and negative \( \frac{d^2y}{dx^2} \)

at \( x = -3 \) (maximum point).

\[ \sqrt{1/2} = 1/2 \] and so \( x = \pm \frac{1}{2} \) therefore \( \frac{d^2y}{dx^2} \)

(d) \( \frac{dy}{dx} = 2 - 1/ \)

therefore the turning point is a 1 \( \sqrt{1/2} \) which is positive at \( x = \frac{1}{2} \)

which is maximum. \( \frac{d^2y}{dx^2} \) minimum, negative at \( x = - \)

(e) \( \frac{dy}{dx} = 2x - 1 \) therefore \( x = 1/2 \) at the turning point.

\[ = 2 \] which is positive therefore the turning point is a minimum. \( \frac{d^2y}{dx^2} \)

= -x(4 + 6x) so at the turning point \( x = 0 \) or \( x = -2/3 \). \( \frac{d^2y}{dx^2} \)

(f) \( \frac{dy}{dx} = -4x - 6 \)

= -4 - 12x which at \( x = 0 \) is negative (maximum) and at \( x = -2/3 \) is \( \frac{d^2y}{dx^2} \)

positive (minimum).

= -8, \( \frac{d^2y}{dx^2} \) so setting this to zero we get 5x3 = -40, therefore \( \frac{d^2y}{dx^2} \)

(g) \( \frac{dy}{dx} = 5 + 40/ \)

\[ x = -2. \]

= -120/x4 which is negative therefore the turning point is a \( \frac{d^2y}{dx^2} \)

maximum.
(h) \( \frac{dy}{dx} = 4x \) therefore \( x = 0 \) at the turning point.

= 4 which is positive therefore the turning point is a minimum.

= -1/2. There are no \( x \) so setting this to zero we get \( x = \frac{-1}{2} \).

(i) \( \frac{dy}{dx} = -1/2 \) turning points for this function. Since any number squared gives a positive answer, we cannot find a value for \( x \) which when squared would give -1/2.

= 4x(2 -x) giving turning points at \( x = 0 \) and \( x = 2 \).

(j) \( \frac{dy}{dx} = 8x - 4 \)

= 8 - 8x which is positive at \( x = 0 \) (minimum) and negative at \( x = 2 \) (maximum).
Section 6.3 – Maximum Total Revenue

Questions

1. At what output, Q, is total revenue maximized if \( TR = 44Q - 2 \)
   What is the maximum value of total revenue? Find the values of total revenue when Q is
   (a) 1 less     (b) 1 more
   than the value that maximizes total revenue.

2. If \( TR = 186Q - 3 \) find the output, Q, that maximizes total revenue.
   Obtain the marginal revenue function and find its value when total revenue is maximized.

3. Demand is given by
   \( P = 240 - 8Q \)
   What is the total revenue function, and at what output is it maximized?
   What is the value of MR at that output?

4. A firm in perfect competition sells all its output at the fixed price of 42.
   Find the firm’s total revenue and marginal revenue functions. Identify any maximum or minimum turning points for these functions.

5. Find the total revenue and marginal revenue functions for a firm with the demand curve \( P = 75/Q \). Identify any maximum or minimum turning points for these functions.

6. For the given demand curves find expressions for marginal revenue, elasticity and total revenue. Show that \( MR = P(1 + 1/E) \) and that maximum
total revenue occurs when \( E = -1 \).

(a) \( P = 72 - 6Q \)

(b) \( P = \frac{16}{Q} \)

Answers

1. \( \frac{dTR}{dQ} = 44 - 4Q \) therefore \( Q = 11 \) at the turning point.

   \[ \frac{dTR}{dQ} = 44 - 4Q \]

   \( \frac{dTR}{dQ} \) = -4 which is negative therefore the turning point is a maximum. \( Q^{\text{ST}} \) TR/dQ

   The maximum value of total revenue occurs when \( Q = 11 \).

   Substitute this in TR to find the maximum value of TR.

   \( TR = 44(11) - 2( \)

   \( 484 - 242 = 242 \). \( \boxed{11} \)

   Maximum TR = 44(11) - 2(440 - 200 = 240). \( \boxed{10} \) (a) When \( Q = 10 \) TR = 44(10) - 2(528 - 288 = 240). \( \boxed{12} \) (b) When \( Q = 12 \) TR = 44(12) - 2(102)

2. \( \frac{dTR}{dQ} = 186 - 6Q \) therefore \( Q = 31 \) at the turning point.

   \[ \frac{dTR}{dQ} = 186 - 6Q \]

   \( \frac{dTR}{dQ} \) = -6 which is negative therefore the turning point is a maximum. \( Q^{\text{ST}} \) TR/dQ

   The maximum value of total revenue occurs when \( Q = 31 \).

   MR = \( \frac{dTR}{dQ} = 186 - 6Q \)

   When \( Q = 31 \) MR = 186 - 6(31) = 0

   MR = 0 when TR is a maximum.

3. \( TR = P.Q = 240Q - 8 \)

   \( \frac{dTR}{dQ} = 240 - 16Q \) therefore \( Q = 15 \) at the turning point.

   \[ \frac{dTR}{dQ} = 240 - 16Q \]

   \( \frac{dTR}{dQ} \) = -16 which is negative therefore the turning point is a maximum. \( Q^{\text{ST}} \) TR/dQ

   maximum.

   MR = \( \frac{dTR}{dQ} = 240 - 16Q \)

   When \( Q = 15 \) MR = 240 - 16(15) = 0

   MR = 0 when TR is a maximum.

4. \( TR = P.Q = 42Q \) which is an upward sloping straight line through the origin.
\[ \frac{dTR}{dQ} = 42 \] and so \( MR = 42 = AR \)

Since \( \frac{dTR}{dQ} \) is a constant it can never equal zero.

This TR function has no turning points because it is a linear function.

\[ \frac{dMR}{dQ} = 0 \] at all values of \( Q \), not at specific turning points. We cannot find a second derivative.

This MR function has no turning points because MR is a constant.

5. \( TR = P.Q = 75 \)

\[ \frac{dTR}{dQ} = 0 \] and so \( MR = 0 \) at all values of \( Q \), not at specific turning points of TR.

We cannot find a second derivative of TR.

This TR function is a constant total revenue function. TR is always the same, so the function has no turning points.

\( MR = 0 \) and so the MR function also has no turning points.

Differentiating the demand curve gives

This could only equal 0 if \( 75 = 0 \), which is not possible. This demand function has no turning points. In the positive quadrant it is a downward sloping curve.

6. (a) \( TR = P.Q = 72Q - 6 \)

\[ \frac{dTR}{dQ} = 72 - 12Q \] therefore \( Q = 6 \) at the turning point.

\[ = -12 \] which is negative therefore the turning point is a maximum.

\( TR \) is a maximum at \( Q = 6 \).

\( MR = 72 - 12Q \)

\( P = 72 - 6Q \) so

\[ \frac{dP}{dQ} = -6 \]

\[ E = \frac{(dQ/dP)(P/Q)}{(-1/6)(72 - 6Q)/Q} = \frac{-(72 - 6Q)}{6Q} \]

\[ P(1 + 1/E) = (72 - 6Q)(1 + [6Q/(72 - 6Q)]) \]
\[(72 - 6Q)(72 - 6Q - 6Q)/(72 - 6Q)\] so

the expression gives \( MR = 72 - 12Q \) as we found before.

At \( Q = 6 \), \( E = -(72 - 6Q)/6Q = -(72 - 36)/36 = -1 \)

(b) \( TR = P \cdot Q = 16 \)

\( dTR/dQ = 0 \) therefore there is no maximum turning point of \( TR \).

\( MR = 0 \)

\( P = 16/Q \) so

\[ \frac{Q}{E} \frac{dP}{dQ} = -16/\]

\[ ) = -1 \frac{Q}{16}\]

\[ E = (dQ/dP)(P/Q) = (\]

\[ P(1 + 1/E) = (16/Q)[1 + (-1)] = 0 \) so the expression gives

\( MR = 0 \) as we found before. The result is true at all values of \( Q \).
Section 6.4 – Maximum Profit

Questions

1. A firm faces a demand function given by
   \[ P = 265 - 10Q \]
   and has the total cost function
   \[ Q^2 + Q^2 \quad TC = 70 + 125Q - 18 \]
   Show at what output profit is maximized, finding price and the maximum value of profit.

2. A firm in perfect competition has demand curve
   \[ P = 94 \]
   and short run total cost curve
   \[ Q^2 + 0.25Q^2 \quad TC = 300 + 190Q - 11 \]
   Find the output at which the firm maximises its profit and the firm’s total revenue, total cost and profit at that output.

3. Find the profit function for a monopoly with the total cost function
   \[ + 360Q + 4,000 Q^2 - 16Q^2 \quad TC = 0.5 \]
   and the demand curve
   \[ P = 1480 - 15Q \]
   At what output is maximum profit achieved? At what price is the output sold? What is the monopolist’s profit?

4. A natural monopoly has the total cost function
for 0 < Q < 1350 \ Q^2 \quad TC = 3830 + 270Q - 0.1

and its demand curve is

\[ AR = 3220 - 6Q \]

Calculate the profit maximizing output and price for the firm and the profit it obtains. Is it producing where MC = MR and MC cuts MR from below?

5. For the monopoly whose demand function and total cost function are given in question 3, what is the effect on the profit maximizing position if taxation is imposed as

(a) a lump sum tax of 168 or

(b) a per unit sales tax of 80?

6. A monopoly has the total cost function

\[ Q^4 + 0.4Q^2 \quad TC = 750 + 520Q - 20 \]

and faces the demand curve

\[ P = 1600 - 20Q \]

(a) Find an expression for the monopolist's profit, and show at what output it is maximized.

(b) What price should the monopolist charge? Check that MC = MR at this price. What profit will the monopolist make?

(c) Suppose now that the government splits up the monopoly and compels the resulting small firms to behave as a perfectly competitive industry. Assume that the total costs of the small firms are the same as those of the monopolist, and that demand is also unchanged. What will be the industry output, price and profit in the short run?

7. A monopoly has the total cost function

\[ Q^4 + Q^2 \quad TC = 10,000 + 300Q - 10 \]

and faces the demand curve

\[ P = 1900 - 20Q \]
(a) Find an expression for the monopolist's profit, and show at what output it is maximized.

(b) What price should the monopolist charge? Check that MC = MR at this price. What profit will the monopolist make?

(c) Suppose now that the government splits up the monopoly and compels the resulting small firms to behave as a perfectly competitive industry. Assume that the total costs of the small firms are the same as those of the monopolist, and that demand is also unchanged. What will be the industry output, price and profit in the short run?

Answers

, so subtracting TC we obtain \( Q^* = TR - TC \) and \( TR = P \cdot Q = 265Q - 10Q \).

\[ Q^* - Q^2 = -70 + 140Q + 8Q \]

so at turning points \( Q^2/dQ = 140 + 16Q - 3 \)

\[ + 16Q + 140 = 0 \quad Q^2 - 3 \]

\[ \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

Using \( Q = \)

\[ Q = \frac{256 + 1680}{44} \]

Only a positive output is meaningful. Check to see if 10 give a maximum turning point.

\[ = 16 - 6Q \]

This equals –44, a negative value, when \( Q = 10 \) so \( Q = 10 \) is a maximum turning point. Substitute to find the price and profit:

\[ P = 265 - 10(10) = 165 = \text{price at profit maximizing position.} \]
\[-70 + 1400 + 800 - 1000 = 1130 \text{ } 10^2 \] 
\[-(10^2) = -70 + 140(10) + 8 \pi \]

\[= TR - TC \text{ and } TR = P.Q = 94Q, \text{ and so subtracting TC we obtain } 2.\]
\[Q^2 - 0.25Q^2 = -300 - 96Q + 11 \pi \]

\[
\frac{d}{dQ} = -96 + 22Q - 0.75 \]
so at turning points (setting equal to 0 and multiplying through by −1)

\[-22Q + 96 = 0 \quad Q = 0.75 \]

\[
Q = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{Using } Q =
\]
\[
(484 - 288)/(1.5) = (22 \pm 14)/(1.5) = 5.33 \text{ or } 24 \sqrt{\pi} \quad Q = [22 \pm
\]

Check to see if one of these values gives a maximum turning point.

\[= -1.5Q \text{ } \frac{d}{dQ} \text{ } d^2 \]
This equals 14, a positive value, when \(Q = 5.33\) and −14, a negative value
when \(Q = 24\) so \(Q = 24\) is a maximum turning point. Substitution gives:

\[TR = 94Q = 94(24) = 2256\]
\[
) = 1980 24^2 + 0.25(24^2) \quad TC = 300 + 190(24) - 11(
\]
\[= TR - TC = 276 Q^2 - 0.25Q^2 = -300 - 96Q + 11 \pi \]

\[Q^2 - 0.5Q^2 = -4000 + 1120Q + \pi \text{ 3.}\]
\[Q = 28 \text{ gives maximum profit}\]
\[P = 1060\]
\[= 17168 \pi \text{ 4.}\]

\[Q^2 = -3830 + 2950Q - 5.9 \pi \text{ 4.}\]
Q = 250 gives maximum profit

\[ P = 1720 \]

\[ 364920 \] 

MC = 270 – 0.2Q = 220 when Q = 250

MR = 3220 – 12Q = 220 when Q = 250 so MC = MR

If Q = 249, MC = 220.2 and MR = 232, so MC cuts MR from below.

5. (a) A lump sum tax of 168 is an addition to fixed cost

Q = 28 still gives maximum profit

\[ P = 1060 \] and profit becomes

\[ 17000 \] 

\[ Q^S - 0.5Q^S = -4000 + 1040Q + \pi' \] (b)

Q = 27.006 now gives maximum profit, price rises to

\[ P = 1074.9 \] and profit falls to

\[ 14967.5 \]

6. (a) \[ + 1080Q - 750, Q = 30 \]

\[ Q^B = -0.4 \]

\[ 20,850 \] (b) \[ P = 1000, MC = MR = 400, \]

(c) Under perfect competition, industry price is determined where MC = AR.

\[ 17,281.78 \] This gives Q = 39.47, P = 810.6,

\[ + 1600Q - 10000, Q = 20 \]

\[ Q^B = -10 \]

\[ 10,000 \] (b) \[ P = 1500, MC = MR = 1100, \]

(c) Under perfect competition, industry price is determined where MC = AR.

\[ 9300.28 \] This gives Q = 23.09, P = 1438.12,
Section 6.5 – Minimum Average Cost

Questions

1. A firm has total costs given by

\[ TC = 0.1 - 6Q^a + 450Q \]

(a) Find the firm’s average total cost, AC, function and show at what output it is at a minimum. What is the minimum value of AC?

(b) Find also marginal cost, MC. What value does it have when AC is a minimum? At what output is MC a minimum?

2. A firm in perfect competition has the short run total cost curve

\[ Q^a + 0.25 Q^{a^2} \]

\[ TC = 70 + 250Q - 12 \]

(a) What is the minimum value of the firm’s average variable cost curve?

(b) What is the firm’s short run supply curve? What is the minimum quantity the firm will supply, and the minimum supply price?

(c) How much will the firm supply at a price of 301? At what output is marginal cost a minimum?
Answers

1. (a) \( AC = \frac{TC}{Q} = 0.1 \)
   
   \[ d \left( AC \right)/dQ = 0.2Q - 6 = 0 \]

   for a possible minimum, so

   \( Q = 30 \) is a turning point. To check for a minimum, find

   \[ = 0.2 \frac{Q}{d} \left( AC \right)/dQ \]

   This is a positive constant, so \( Q = 30 \) is a minimum.

   Substituting \( Q = 30 \)

   \[ ) - 6(30) + 450 = 360 \]

   (b) \( MC = \frac{d(TC)}{dQ} = 0.3 \)

   At \( Q = 30 \) we have

   \[ ) - 12Q + 450 \frac{Q}{d} = 360 = \text{AC} \]

   \[ = 0.3( \]

   \[ d(MC)/dQ = 0.6Q - 12 = 0 \] at a turning point.

   \( Q = 20 \) and

   \[ = 0.6 \] which is positive, so \( Q = 20 \) is a minimum. \( Q = d(MC)/dQ \)

2. (a) \( AVC = \frac{VC}{Q} = 250 - 12Q + 0.25 \)

   \[ d \left( AVC \right)/dQ = -12 + 0.5Q = 0 \]

   for a possible minimum, so

   \( Q = 24 \) is a turning point. To check for a minimum, find

   \[ = 0.5 \frac{Q}{d} \left( AVC \right)/dQ \]

   This is a positive constant, so \( Q = 24 \) is a minimum.

   Substituting \( Q = 24 \) gives the minimum value of the firm’s average

   variable cost curve:

   \[ ) = 106 \frac{2Q}{d} \left( AVC \right)/dQ = 250 - 12(24) + 0.25( \]

   (b) The firm’s short run supply curve is its marginal cost curve above the

   intersection with the average variable cost curve.
MC = d (TC)/dQ = 250 – 24Q + 0.75

The firm’s short run supply curve is

\[ P = 250 - 24Q + 0.75 \]

The minimum quantity the firm will supply is 24 and the minimum supply price is 106.

(c) If \( P = 301 \), solve for \( Q \)

\[ Q = 301 = 250 - 24Q + 0.75 \]

\[ -24Q - 51 = 0 \]

Factorise or use the formula to find

\[ Q = 34 \text{ or } Q = -2 \]

The minimum quantity the firm will supply is 24. \( Q = 34 \) exceeds this, so the firm will supply 34 at a price of 301.

\[ d\ (MC)/dQ = -24 + 1.5Q \]

\[ Q = 16 \] at a turning point.

\[ = 1.5 \] which is positive, so \( Q = 16 \) gives minimum MC. \( Q^* \) (MC)/dQ

Section 6.6 – Average and Marginal

Product of Labour

Questions

1. A firm’s short-run production function is given by

\[ Q = 180 \]

Find the marginal and average products of labour.

Check that when average product is maximized, APL=MPL.
2. The output a firm produces per week is given by its short run production function

\[ Q = 91L + 16 \]

where \( L \) is the number of persons employed.

(a) Find the average product of labour and show at what number of employees it reaches a maximum.

(b) Obtain an expression for the marginal product of labour. Over what range of number of employees do diminishing marginal returns occur?

(c) What is the maximum value of the average product of labour? What is the value of the marginal product of labour at that point?

3. For a firm with the short-run production function

\[ Q = 42L + 27 \]

find APL and show at what number of employees it reaches a maximum.

Find also MPL and show when there are diminishing marginal returns.

Answers

1. MPL = \( \frac{dQ}{dL} = 360L - 9 \)

APL = \( \frac{Q}{L} = 180L - 3 \)

\( \frac{d (APL)}{dL} = 180 - 6L = 0 \) (for a maximum)

So, \( L = 30 \) is a turning point. The second derivative is

\[ = - 6 \frac{L^2}{L} \frac{(APL)}{dL} \]

The negative sign indicates a maximum, so APL is a maximum at \( L = 6 \).

Substituting in APL and MPL we find

\[ 2700 \times 30 \times 30 \times 3 \]

\[ APL = 180(30) - 3( \]

\[-6\times30^2 \]
\[ MPL = 360(30) - 9 \]

APL and MPL intersect at the maximum value of APL.

1. \[ MPL = 360(30) - 9 \]

\( L \)

(a) APL = 91 + 16L

\[ d (APL)/dL = 16 - 2L = 0 \]

This is negative, so L = 8 is a maximum.

(b) MPL = 91 + 32L - 3

\[ d (MPL)/dL = 32 - 6L = 0 \]

This is negative, so L = 8 is a maximum.

After the maximum MPL falls, or diminishes, so diminishing marginal returns occur when more than 5.33 people are employed (or when 6 or more people are employed if all employees work full time).

(c) Substitute L = 8

\[ APL = 91 + 16(8) - ( \]

\[ MPL = 91 + 32(8) - 3 \]

2. APL = 42 + 27L - 0.5

\[ d (APL)/dL = 27 - L = 0 \]

This is negative, so L = 27 is a maximum.

MPL = 42 + 54L - 1.5

\[ d (MPL)/dL = 54 - 3L = 0 \]

This is negative, so L = 18 is a maximum.

Diminishing marginal returns occur when more than 18 people are employed.
Section 6.7 – Tax Rate which Maximizes Tax Revenue

Questions

1. Choose the per unit rate of tax, $t$, which maximizes tax revenue in the market described by the equations

   Demand: $P = 212 - 5Q$
   
   Supply: $P = 8 + 12Q$

assuming that the market reaches equilibrium.
Answers

1. After tax supply is given by: \( P - t = 8 + 12Q \) or
   \[
P = 8 + 12Q + t
   \]
   Equate the expressions for \( P \) from the after tax supply and demand equations.
   \[
   8 + 12Q + t = 212 - 5Q, \text{ which gives}
   \]
   \[
   Q = 12 - t/17
   \]
   This expression for \( Q \) assumes market equilibrium is achieved.
   
   \[ t.Q = \text{tax revenue} \]
   
   Substitute for \( Q \)
   \[
   t.Q = 12t - \]
   
   We want to choose the value of \( t \) that maximizes \( t.Q \). To do this we differentiate with respect to \( t \) and set equal to 0 for a maximum
   
   \[
   d (t.Q)/dt = 12 -2t/17 = 0
   \]
   
   \[
   t = 102
   \]
   
   \[
   = -2/17 (t.Q)/d t
   \]
   
   This is negative, so we have a maximum, therefore \( t = 102 \) is the per unit tax that maximizes tax revenue.
Section 6.8 – Minimizing Total Inventory

Costs

Questions

1. A computer manufacturer uses 1170 video boards per year. These are bought in from a specialist supplier at a cost of $200 each. Placing an order for video boards costs $20 in phone calls, paperwork and carriage. The computer manufacturer estimates the cost of holding video boards in stock to be $1 per board per week. To minimize the total inventory costs for video boards, how many should be ordered at one time? How many orders for video boards will be placed each year?

2. A company buys components from a supplier at $240 each, and uses them at the rate of 6,000 per year. Each order placed costs $80, and orders are delivered immediately. To hold one component in stock for one year costs 40% of its value. To minimize the total inventory costs for these components, how many should be ordered at one time? How many orders should be placed each year?
Answers

1. Choose $Q$ to minimize $\frac{23400}{Q} + 26Q$. This gives
   
   $EOQ = 30$, which is the quantity that should be ordered.

   39 orders are placed each year.

2. Choose $Q$ to minimize $\frac{480000}{Q} + 48Q$. This gives
   
   $EOQ = 100$, which is the quantity that should be ordered.

   60 orders are placed each year.
Section 7.2 – The Chain Rule

In some situations, it is of course easier to multiply out an expression and differentiate rather than use the chain rule. However, it is suggested that you use the chain rule to answer the following questions to give you practice at recognizing how to use it effectively.

Differentiate with respect to x

a) \( y = (7 + 5x)^2 \)  
b) \( y = (9x - 10)^2 \)

c) \( y = \frac{5}{7x + 3} \)  
d) \( y = \sqrt{8x - 6} \)

e) \( y = (3 - 11x)^2 \)  
f) \( y = \frac{1}{x + 8} \)

g) \( y = \left( 5x^3 - 6 \right)^2 \)  
h) \( y = (x^8 - 2x^2 + 7x)^2 \)

i) \( y = \sqrt{x^8 - 9} \)  
j) \( y = \frac{1}{3x^2 + 8x^2} \)

k) \( y = \frac{2x}{7x^2 + 1} \)  
l) \( y = \left( 12x^2 - 4x - 3 \right)^2 \)

m) \( y = \frac{2x}{1 - 0.75(1 - x)} \)  
n) \( y = \left( 5x + 9 \right)^3 \)
o) \( y = \sqrt{8x^2 - 3x - 1} \)  

p) \( y = \frac{10x}{1 - 0.8(1 - x)} \)

q) \( y = (x^4 - 3x^3 - 7)^2 \)  
r) \( y = (x^4 - 2x - 8)^3 \)

s) \( y = \frac{1}{\sqrt{x^2 - 6x}} \)  
t) \( y = \frac{1}{(x^2 + 5x)^2} \)

u) \( y = \left(4x^2 - xy \right)^{0.2} \)  
v) \( y = (x^2 + 9)^4 \)

w) \( y = \left(6x^2 + 5x - 1 \right)^{0.7} \)

Answers

a) \( 10(7 + 5x) \)  
b) \( 18(9x - 10) \)

c) \( \frac{-35}{(7x + 3)^5} \)  
d) \( \frac{4}{5x - 1} \)

e) \( -22(3 - 11x) \)  
f) \( \frac{-1}{(x + 8)^3} \)

g) \( 30x^7 \) \( (5x^7 - 3) - 6 \)  
h) \( 2(x^6 - 2x^2 + 7x)(3x^6 - 4x + 7) \)
i) \( 0.5(9^x - 9)^{-1/2} \):
\[
\frac{3 + 16x}{(3x + 8x^2)^2}
\]

j) \( -\frac{2x^2 - 9}{2\sqrt{x^2 - 9x}} \):
\[
\frac{2x^2 - 9}{2\sqrt{x^2 - 9x}}
\]

k) \( \frac{20(14x)}{(7x^2 + 1)^3} \):
\[
\frac{20(14x)}{(7x^2 + 1)^3} = \frac{280x}{(7x^2 + 1)^3}
\]

l) \( \frac{280x}{(7x^2 + 1)^3} \):
\[
\frac{280x}{(7x^2 + 1)^3}
\]

m) \( \frac{2x}{1 - 0.75 + 0.75x} \):
\[
\frac{24}{0.25 + 0.75x}
\]

n) \( \frac{24}{1 - 0.75 + 0.75x} \):
\[
\frac{24}{1 - 0.75 + 0.75x}
\]

o) \( \frac{16x - 3}{2\sqrt{8x^2 - 8x - 1}} \):
\[
\frac{16x - 3}{2\sqrt{8x^2 - 8x - 1}}
\]

p) \( \frac{10x}{0.2 + 0.8x} \):
\[
\frac{80}{0.2 + 0.8x}
\]

q) \( 2(5x^2 - 7)(4x^3 - 10x) \):
\[
2(5x^2 - 7)(4x^3 - 10x)
\]

r) \( 3(x^2 + 2x - 6)^2 \):
\[
3(x^2 + 2x - 6)^2
\]

s) \( -2x^2 - 3x^2 - 2x^2 \):
\[
\frac{(3x^2 - 6)}{2}
\]

\[
\frac{(3x^2 - 6)}{2}
\]

\[
2x + 5
\]

\[
(2x + 5)
\]

\[
-6x^2 + 5x^3
\]

\[
(2x + 5)
\]

\[
-6x^2 + 5x^3
\]

\[
\frac{8x - 9}{4(4x^2 - 9x)^{0.75}}
\]

\[
\frac{8x - 9}{4(4x^2 - 9x)^{0.75}}
\]

\[
4(x^2 + 9)^3(2x) = 8x(x^2 + 9)^3
\]

\[
4(x^2 + 9)^3(2x) = 8x(x^2 + 9)^3
\]
Section 9.3 – Lagrange Multiplier Method

Questions

1. A consumer has an income of $420 and the utility function

\[ U = 10x^0.5 y^0.5 \]

Goods X and Y sell at prices of $7 and $14 respectively. Find what quantities of X and Y maximize the consumer’s utility. Show that when the consumer buys the optimal quantities of X and Y, \( M U_x / M U_y = P_x / P_y \).

2. Opera tickets cost $32 and cinema tickets cost $4.50. A young graduate budgets $287 to spend on these entertainments in the next year. If his utility function is

\[ U = 8R + 30C + 5RC - R^2 - 2C^2 \]

what numbers of opera tickets, R, and cinema tickets, C, should he buy to maximize his utility?

3. A consumer spends her income of $1280 on goods X and Y which sell at prices of $64 and $16 respectively. If her utility function is

\[ U = X^0.5 Y^{0.3} \]
show what quantities of the goods maximize her utility. What is the value of her utility when she buys these quantities? How much of her income does she spend on each good?

4. A firm has the production function

\[ Q = L^\frac{1}{2}K^\frac{1}{2} \]

where L and K are the units of labour and capital used in production.

(a) If the firm budgets $720 for purchasing inputs at prices of $5 for labour and $20 for capital, show what quantities it should buy of each to maximise its output. What is the output produced?

(b) Compare the input combination \( K = 16, L = 83 \) with that chosen in (a). Can this new combination be achieved with the budget available, and what output would it produce?

Answers

1. \( X = 30, Y = 15 \). \( MU_X = 3.535, MU_Y = 7.0711, MU_X/MU_Y = 0.5 = P_X/P_Y \)

2. \( R = 7, C = 14 \)

3. \( X = 10, Y = 40, U = 20 \), $640 is spent on each good.

4. (a) \( K = 12, L = 96, Q = 18.24 \).

(b) Combination costs $735, so is not affordable. The output produced is 18.22, slightly less than the output with the optimal affordable combination of factors.