Chapter 5
Discrete
Probability Distributions

Chapter Goals

After completing this chapter, you should be able to:

- Calculate and interpret the expected value of a probability distribution
- Apply the binomial distribution to applied problems
- Compute probabilities for the Poisson and hypergeometric distributions
- Recognize when to apply discrete probability distributions
Introduction to Probability Distributions

- Random Variable
  - Represents a possible numerical value from a random event
  - Takes on different values based on chance

Discrete Random Variable

- A discrete random variable is a variable that can assume only a countable number of values
  - Many possible outcomes:
    - number of complaints per day
    - number of TV’s in a household
    - number of rings before the phone is answered
  - Only two possible outcomes:
    - gender: male or female
    - defective: yes or no
    - taste: sweet or sour
Continuous Random Variable

- A **continuous random variable** is a variable that can assume any value on a continuum (can assume an uncountable number of values)
  - thickness of an item
  - time required to complete a task
  - temperature of a solution
  - height, in inches

- These can potentially take on any value, depending only on the ability to measure accurately.

Discrete Random Variables

- Can only assume a countable number of values

Examples:

- **Roll a die twice**
  - Let \( x \) be the number of times 4 comes up
  - (then \( x \) could be 0, 1, or 2 times)

- **Toss a coin 5 times.**
  - Let \( x \) be the number of heads
  - (then \( x \) = 0, 1, 2, 3, 4, or 5)
Discrete Probability Distribution

Experiment: Toss 2 Coins. Let \( x = \# \text{ heads.} \)

- **4 possible outcomes**
  - H H
  - H T
  - T H
  - T T

**Probability Distribution**

<table>
<thead>
<tr>
<th>( x ) Value</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1/4 = 0.25</td>
</tr>
<tr>
<td>1</td>
<td>2/4 = 0.50</td>
</tr>
<tr>
<td>2</td>
<td>1/4 = 0.25</td>
</tr>
</tbody>
</table>

Discrete Probability Distribution

- A list of all possible \([ x_i, P(x_i) ]\) pairs
  - \( x_i = \) Value of Random Variable (Outcome)
  - \( P(x_i) = \) Probability Associated with Value
- \( x_i \)'s are mutually exclusive
  - (no overlap)
- \( x_i \)'s are collectively exhaustive
  - (nothing left out)
- \( 0 \leq P(x_i) \leq 1 \) for each \( x_i \)
- \( \sum P(x_i) = 1 \)
Discrete Probability Distribution

Example: For a survey conducted by local chamber of commerce to determine number of PC’s owned by a family, write the probability distribution of the PC’s owned by a family.

<table>
<thead>
<tr>
<th># of PC’s owned</th>
<th>Frequency</th>
<th>Relative Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>120</td>
<td>.12</td>
</tr>
<tr>
<td>1</td>
<td>180</td>
<td>.18</td>
</tr>
<tr>
<td>2</td>
<td>470</td>
<td>.47</td>
</tr>
<tr>
<td>3</td>
<td>230</td>
<td>.23</td>
</tr>
<tr>
<td></td>
<td><strong>N = 1000</strong></td>
<td><strong>Sum = 1.000</strong></td>
</tr>
</tbody>
</table>

Discrete Probability Distribution

- Let $x$ denote the number of PCs owned by a family.
- Then $x$ can take any of the four possible values (0, 1, 2, and 3).

<table>
<thead>
<tr>
<th># of PC’s owned</th>
<th>Frequency</th>
<th>Relative Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>120</td>
<td>.12</td>
</tr>
<tr>
<td>1</td>
<td>180</td>
<td>.18</td>
</tr>
<tr>
<td>2</td>
<td>470</td>
<td>.47</td>
</tr>
<tr>
<td>3</td>
<td>230</td>
<td>.23</td>
</tr>
<tr>
<td></td>
<td><strong>N = 1000</strong></td>
<td><strong>Sum = 1.000</strong></td>
</tr>
</tbody>
</table>
Discrete Probability Distribution

**Example:** For the following table

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

a) Construct a probability distribution table

b) Find the following probabilities

i. $P(x = 3)$  
ii. $P(x < 4)$  
iii. $P(x \geq 3)$  
iv. $P(2 \leq x \leq 4)$

---

Discrete Random Variable Summary Measures

- **Expected Value** of a discrete distribution  
  (Weighted Average)

$$E(x) = \sum xP(x)$$

**Example:** Toss 2 coins,  
$x = \#$ of heads,  
compute expected value of $x$:

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(x)$</td>
<td>.25</td>
<td>.50</td>
<td>.25</td>
</tr>
</tbody>
</table>
**Discrete Random Variable Summary Measures**

(continued)

- **Standard Deviation** of a discrete distribution

\[
\sigma_x = \sqrt{\sum (x - E(x))^2 P(x)}
\]

where:

- \(E(x)\) = Expected value of the random variable
- \(x\) = Values of the random variable
- \(P(x)\) = Probability of the random variable having the value of \(x\)

---

**Example:** Toss 2 coins, \(x\) = # heads, compute standard deviation (recall \(E(x) = 1\))

\[
\sigma_x = \sqrt{\sum (x - E(x))^2 P(x)}
\]
Example: Recall “number of PC’s owned by a family” example. Find the mean number of PCs owned by a family and their variance.

Mean:

<table>
<thead>
<tr>
<th>x</th>
<th>P(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>.12</td>
</tr>
<tr>
<td>1</td>
<td>.18</td>
</tr>
<tr>
<td>2</td>
<td>.47</td>
</tr>
<tr>
<td>3</td>
<td>.23</td>
</tr>
</tbody>
</table>

We need to find \( x.p(x) \) for each value of \( x \) and then add them up together.

Variance & Standard Deviation:

Here we need to find two columns \( x.p(x) \) and \( x^2.p(x) \)

<table>
<thead>
<tr>
<th>x</th>
<th>P(x)</th>
<th>x.P(x)</th>
<th>x^2.P(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>.12</td>
<td>0(.12) = 0.00</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>.18</td>
<td>1(.18) = 0.18</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>.47</td>
<td>2(.47) = 0.94</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>.23</td>
<td>3(.23) = 0.69</td>
<td></td>
</tr>
</tbody>
</table>
Probability Distributions

- Discrete Probability Distributions
  - Binomial
  - Hypergeometric
  - Poisson

- Continuous Probability Distributions
  - Normal
  - Uniform
  - Exponential

The Binomial Distribution

- Discrete Probability Distributions
  - Binomial
  - Poisson
  - Hypergeometric
The Binomial Distribution

- Characteristics of the Binomial Distribution:
  - A trial has only two possible outcomes – “success” or “failure”
  - There is a fixed number, n, of identical trials
  - The trials of the experiment are independent of each other
  - The probability of a success, p, remains constant from trial to trial
  - If p represents the probability of a success, then (1-p) = q is the probability of a failure

Binomial Distribution Settings

- A manufacturing plant labels items as either defective or acceptable
- A firm bidding for a contract will either get the contract or not
- A marketing research firm receives survey responses of “yes I will buy” or “no I will not”
- New job applicants either accept the offer or reject it
Counting Rule for Combinations

- **Recall:** combination is an outcome of an experiment where $x$ objects are selected from a group of $n$ objects

\[
C^n_x = \frac{n!}{x!(n-x)!}
\]

where:
- $C^n_x$ = number of combinations of $x$ objects selected from $n$ objects
- $n! = n(n-1)(n-2) \ldots (2)(1)$
- $x! = x(x-1)(x-2) \ldots (2)(1)$
- $0! = 1$ (by definition)

Binomial Distribution Formula

\[
P(x) = \frac{n!}{x!(n-x)!} p^x q^{n-x}
\]

$P(x) =$ probability of $x$ successes in $n$ trials, with probability of success $p$ on each trial

- $x =$ number of ‘successes’ in sample, $(x = 0, 1, 2, \ldots, n)$
- $p =$ probability of “success” per trial
- $q =$ probability of “failure” = $(1 - p)$
- $n =$ number of trials (sample size)
Binomial Distribution

Characteristics

- Mean

\[ \mu = E(x) = np \]

- Variance and Standard Deviation

\[ \sigma^2 = npq \]
\[ \sigma = \sqrt{npq} \]

Where

- \( n \) = sample size
- \( p \) = probability of success
- \( q = (1 - p) \) = probability of failure

Examples

<table>
<thead>
<tr>
<th>X</th>
<th>P(X) n = 5 p = 0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.2</td>
</tr>
<tr>
<td>1</td>
<td>0.4</td>
</tr>
<tr>
<td>2</td>
<td>0.2</td>
</tr>
<tr>
<td>3</td>
<td>0.1</td>
</tr>
<tr>
<td>4</td>
<td>0.06</td>
</tr>
<tr>
<td>5</td>
<td>0.01</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>X</th>
<th>P(X) n = 5 p = 0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.3</td>
</tr>
<tr>
<td>1</td>
<td>0.3</td>
</tr>
<tr>
<td>2</td>
<td>0.2</td>
</tr>
<tr>
<td>3</td>
<td>0.15</td>
</tr>
<tr>
<td>4</td>
<td>0.1</td>
</tr>
<tr>
<td>5</td>
<td>0.05</td>
</tr>
</tbody>
</table>
Binomial Distribution Example

Example: 35% of all voters support Proposition A. If a random sample of 10 voters is polled, what is the probability that exactly three of them support the proposition? What is the mean and variance of those voters who support Proposition A?

Using Binomial Tables

<table>
<thead>
<tr>
<th>n = 10</th>
<th>x</th>
<th>p=.15</th>
<th>p=.20</th>
<th>p=.25</th>
<th>p=.30</th>
<th>p=.35</th>
<th>p=.40</th>
<th>p=.45</th>
<th>p=.50</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.1969</td>
<td>0.1074</td>
<td>0.0563</td>
<td>0.0282</td>
<td>0.0135</td>
<td>0.0060</td>
<td>0.0025</td>
<td>0.0010</td>
<td>10</td>
</tr>
<tr>
<td>1</td>
<td>0.3474</td>
<td>0.2684</td>
<td>0.1877</td>
<td>0.1211</td>
<td>0.0725</td>
<td>0.0403</td>
<td>0.0207</td>
<td>0.0098</td>
<td>9</td>
</tr>
<tr>
<td>2</td>
<td>0.2759</td>
<td>0.3020</td>
<td>0.2816</td>
<td>0.2335</td>
<td>0.1757</td>
<td>0.1209</td>
<td>0.0763</td>
<td>0.0439</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>0.1298</td>
<td>0.2013</td>
<td>0.2503</td>
<td>0.2668</td>
<td>0.2522</td>
<td>0.2150</td>
<td>0.1665</td>
<td>0.1172</td>
<td>7</td>
</tr>
<tr>
<td>4</td>
<td>0.0401</td>
<td>0.0881</td>
<td>0.1460</td>
<td>0.2001</td>
<td>0.2377</td>
<td>0.2508</td>
<td>0.2384</td>
<td>0.2051</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>0.0085</td>
<td>0.0264</td>
<td>0.0564</td>
<td>0.1029</td>
<td>0.1536</td>
<td>0.2007</td>
<td>0.2340</td>
<td>0.2461</td>
<td>5</td>
</tr>
<tr>
<td>6</td>
<td>0.0012</td>
<td>0.0055</td>
<td>0.0162</td>
<td>0.0368</td>
<td>0.0689</td>
<td>0.1115</td>
<td>0.1596</td>
<td>0.2051</td>
<td>4</td>
</tr>
<tr>
<td>7</td>
<td>0.0001</td>
<td>0.0008</td>
<td>0.0031</td>
<td>0.0090</td>
<td>0.0212</td>
<td>0.0425</td>
<td>0.0746</td>
<td>0.1172</td>
<td>3</td>
</tr>
<tr>
<td>8</td>
<td>0.0000</td>
<td>0.0001</td>
<td><strong>0.0004</strong></td>
<td>0.0014</td>
<td>0.0043</td>
<td>0.0106</td>
<td>0.0229</td>
<td>0.0439</td>
<td>2</td>
</tr>
<tr>
<td>9</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0001</td>
<td>0.0005</td>
<td>0.0016</td>
<td>0.0042</td>
<td>0.0098</td>
<td>1</td>
</tr>
<tr>
<td>10</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0</td>
</tr>
</tbody>
</table>

Examples:
Binomial Distribution Example

Example: Based on data from A peter D. Hart Research Associates’ poll on consumer buying habits and attitudes, it was estimated that 5% of American shoppers are status shoppers, that is, shoppers who love to buy designer labels. A random sample of eight American shoppers is selected. Using Binomial Table, answer the following:

- Find the probability that exactly 3 shoppers are status shoppers.
- Find the probability that at most 2 shoppers are status shoppers.

Binomial Distribution Example

- Find the probability that at least 3 shoppers are status shoppers.
- Let \( x \) be the number of status shoppers, Write the probability distribution of \( x \) and draw a graph of this probability.
Shape of the Binomial Distribution and the probability of Success

- For any number of trials $n$:
  - The binomial probability distribution is symmetric if $p = .5$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$x$</th>
<th>.05</th>
<th>.1</th>
<th>...</th>
<th>.5</th>
<th>...</th>
<th>.9</th>
<th>.95</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0</td>
<td>.8145</td>
<td>.6561</td>
<td>...</td>
<td>.0625</td>
<td>...</td>
<td>.0001</td>
<td>.0000</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>.1715</td>
<td>.2916</td>
<td>...</td>
<td>.2500</td>
<td>...</td>
<td>.0036</td>
<td>.0005</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>.0135</td>
<td>.0486</td>
<td>...</td>
<td>.3750</td>
<td>...</td>
<td>.0486</td>
<td>.0135</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>.0005</td>
<td>.0036</td>
<td>...</td>
<td>.2500</td>
<td>...</td>
<td>.2916</td>
<td>.1715</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>.0000</td>
<td>.0001</td>
<td>...</td>
<td>.0625</td>
<td>...</td>
<td>.6561</td>
<td>.8145</td>
</tr>
</tbody>
</table>

Shape of the Binomial Distribution and the probability of Success

- For any number of trials $n$:
  - The binomial probability distribution is right skewed if $p < .5$
Shape of the Binomial Distribution and the probability of Success

- For any number of trials $n$:
  - The binomial probability distribution is left skewed if $p > 0.5$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$x$</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0</td>
<td>.8145 .6561 ... .0625 ... .0001 .0000</td>
</tr>
<tr>
<td>1</td>
<td>.1715 .2916 ... .2500 ... .0036 .0005</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>.0135 .0486 ... .3750 ... .0486 .0135</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>.0005 .0036 ... .2500 ... .2916 .1715</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>.0000 .0001 ... .0625 ... .6561 .8145</td>
<td></td>
</tr>
</tbody>
</table>

The Hypergeometric Distribution

Discrete Probability Distributions

- Binomial
- Hypergeometric
- Poisson
The Hypergeometric Distribution

- “n” trials in a sample taken from a finite population of size N
- Sample taken without replacement
- Trials are dependent
- Concerned with finding the probability of “x” successes in the sample where there are “X” successes in the population

Hypergeometric Distribution Formula

(Two possible outcomes per trial: success or failure)

\[ P(x) = \frac{C_{N-X}^{n-x} \cdot C_x^x}{C_N^n} \]

Where
- \( N \) = population size
- \( X \) = number of successes in the population
- \( n \) = sample size
- \( x \) = number of successes in the sample
- \( n - x \) = number of failures in the sample
Hypergeometric Distribution

Example

Example: 3 Light bulbs were selected from 10. Of the 10 there were 4 defective. What is the probability that 2 of the 3 selected are defective?

The Poisson Distribution
The Poisson Distribution

- Characteristics of the Poisson Distribution:
  - The outcomes of interest are rare relative to the possible outcomes
  - The average number of outcomes of interest per time or space interval is $\lambda$
  - The number of outcomes of interest are random, and the occurrence of one outcome does not influence the chances of another outcome of interest
  - The probability that an outcome of interest occurs in a given segment is the same for all segments

Poisson Distribution Formula

$$P(x) = \frac{(\lambda t)^x e^{-\lambda t}}{x!}$$

where:
- $t$ = size of the segment of interest
- $x$ = number of successes in segment of interest
- $\lambda$ = expected number of successes in a segment of unit size
- $e$ = base of the natural logarithm system (2.71828...)
Poisson Distribution Characteristics

- Mean
  \[ \mu = \lambda t \]

- Variance and Standard Deviation
  \[ \sigma^2 = \lambda t \]
  \[ \sigma = \sqrt{\lambda t} \]

where \( \lambda \) = number of successes in a segment of unit size
\( t \) = the size of the segment of interest

Using Poisson Tables

<table>
<thead>
<tr>
<th>( x )</th>
<th>0.10</th>
<th>0.20</th>
<th>0.30</th>
<th>0.40</th>
<th>0.50</th>
<th>0.60</th>
<th>0.70</th>
<th>0.80</th>
<th>0.90</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.9048</td>
<td>0.8187</td>
<td>0.7408</td>
<td>0.6703</td>
<td>0.6065</td>
<td>0.5488</td>
<td>0.4966</td>
<td>0.4493</td>
<td>0.4066</td>
</tr>
<tr>
<td>1</td>
<td>0.0905</td>
<td>0.1637</td>
<td>0.2222</td>
<td>0.2681</td>
<td>0.3033</td>
<td>0.3293</td>
<td>0.3476</td>
<td>0.3595</td>
<td>0.3659</td>
</tr>
<tr>
<td>2</td>
<td>0.0045</td>
<td>0.0164</td>
<td>0.0333</td>
<td>0.0536</td>
<td>0.0758</td>
<td>0.0988</td>
<td>0.1217</td>
<td>0.1438</td>
<td>0.1647</td>
</tr>
<tr>
<td>3</td>
<td>0.0002</td>
<td>0.0011</td>
<td>0.0033</td>
<td>0.0072</td>
<td>0.0126</td>
<td>0.0198</td>
<td>0.0284</td>
<td>0.0386</td>
<td>0.0494</td>
</tr>
<tr>
<td>4</td>
<td>0.0000</td>
<td>0.0001</td>
<td>0.0003</td>
<td>0.0007</td>
<td>0.0016</td>
<td>0.0030</td>
<td>0.0050</td>
<td>0.0077</td>
<td>0.0111</td>
</tr>
<tr>
<td>5</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0001</td>
<td>0.0002</td>
<td>0.0004</td>
<td>0.0007</td>
<td>0.0012</td>
<td>0.0020</td>
</tr>
<tr>
<td>6</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0003</td>
</tr>
<tr>
<td>7</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
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Example: Find \( P(x = 2) \) if \( \lambda = 0.05 \) and \( t = 100 \)
Chapter Summary

- Reviewed key discrete distributions
  - binomial
  - Poisson
  - hypergeometric
- Found probabilities using formulas and tables
- Recognized when to apply different distributions
- Applied distributions to decision problems

Problems

Over a long period of time it has been observed that a given sniper can hit a target on a single trial with a probability = .8. Suppose he fires four shots at the target.

a) What is the probability that he will hit the target exactly two times?

b) What is the probability that he will hit the target at least once?
Problems

Five percent of all VCRs manufactured by a large factory are defective. A quality control inspector selects three VCRs from the production line. What is the probability that exactly one of these three VCRs is defective?

Problems

Dawn corporation has 12 employees who hold managerial positions. Of them, 7 are female and 5 are male. The company is planning to send 3 of these 12 managers to a conference. If 3 managers are randomly selected out of 12,

a) find the probability that all 3 of them are female

b) find the probability that at most 1 of them is a female
Problems

A case of soda has 12 bottles, 3 of which contain diet soda. A sample of 4 bottles is randomly selected from the case

a) find the probability distribution of $x$, the number of diet sodas in the sample

b) what are the mean and variance of $x$?

Problems

The average number of traffic accidents on a certain section of highway is two per week. Assume that the number of accidents follows a Poisson distribution with $\lambda = 2$.

a) Find the probability of no accidents on this section of highway during a 1-week period.

b) Find the probability of at most three accidents on this section of highway during a 2-week period.
On average, two new accounts are opened per day at an Imperial Savings Bank branches. Using Tables, find the probability that on a given day the number of new accounts opened at this bank will be

a) Exactly 6  
b) At most 3  
c) At least 7

d) Mean & Variance

The materials of this presentation were mostly taken from the PowerPoint files accompanied Business Statistics: A Decision-Making Approach, 7e © 2008 Prentice-Hall, Inc.